A Note on a Vector-Variate Normal Distribution and a Stationary Autoregressive Process

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It is shown that weak stationarity of a first-order autoregressive process implies that eigenvalues of the coefficient matrix are less than 1 in absolute value.

Nguyen (1997) has shown (Theorem 2.1) that if $X_1$ and $X_2$ are identically distributed random vectors such that

$$X_2 = BX_1 + U_2,$$  \hspace{1cm} (1)

$U_2$ and $X_1$ are independent, and $U_2$ has the distribution $N(0, \Sigma)$ with $\Sigma$ positive definite, then (a) the eigenvalues of $B$ have modulus less than 1 and (b) $X_1$ and $X_2$ have a joint normal distribution with covariance matrix

$$E[X_1X_2'] = (\Gamma B \Gamma'),$$  \hspace{1cm} (2)

where

$$\Gamma = \sum_{s=0}^{\infty} B^s \Sigma B'^s.$$  \hspace{1cm} (3)

If the result is stated in the form of

$$X_t = BX_{t-1} + U_t,$$  \hspace{1cm} (4)
for \( t = 2 \), it may be recognized as a form of the statement that a strictly stationary (autoregressive) process defined by (4) implies that the eigenvalues of \( \mathbf{B} \) are less than 1 in absolute value and that if \( \mathbf{U}_t \) is normal

\[
\mathbf{X}_t = \sum_{s=0}^{\infty} \mathbf{B}^s \mathbf{U}_{t-s}
\]  

(5)
is Gaussian.

The purpose of this note is to show in a simple way that only stationarity in the wide sense needed for conclusion (a).

**Theorem.** Let \( \mathbf{X}_1, \mathbf{X}_2, \) and \( \mathbf{U}_2 \) be related by (1) with \( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) having the common covariance matrix \( \mathbf{\Gamma} \), \( \mathbf{U}_2 \) having a nonsingular covariance matrix \( \mathbf{\Sigma} \), and \( \mathbf{X}_1 \) and \( \mathbf{U}_2 \) uncorrelated. Then the eigenvalues of \( \mathbf{B} \) are less than 1 in absolute value.

**Proof.** An eigenvalue \( \lambda \) and eigenvector \( \mathbf{x} \) satisfy

\[
\mathbf{B} \mathbf{x} = \lambda \mathbf{x}. 
\]  

(6)

Then \( \mathbf{\Gamma} = \mathbf{B} \mathbf{B}^\prime + \mathbf{\Sigma} \) implies

\[
\mathbf{x}^\prime \mathbf{\Sigma} \mathbf{x} = |\lambda|^2 \mathbf{x}^\prime \mathbf{\Gamma} \mathbf{x} + \mathbf{x}^\prime \mathbf{x}. 
\]  

(7)

Since \( \mathbf{x}^\prime \mathbf{\Sigma} \mathbf{x} > 0 \), (7) implies \( \mathbf{x}^\prime \mathbf{\Gamma} \mathbf{x} > 0 \) and \( |\lambda|^2 < 1 \).

A sequence of random vectors \( \mathbf{X}_t \) can be constructed recursively by (4), \( t = 3, \ldots \). A consequence of the theorem is that (5) converges in the mean and \( \{ \mathbf{X}_t \} \) is stationary; if the \( \mathbf{U}_t \) is independent of the \( \mathbf{X}_{t-1} \), then \( \{ \mathbf{X}_t \} \) is Gaussian. See, for example, Anderson (1971, p. 179).

If \( \mathbf{X}_t \) has mean \( \mathbf{\mu}_t = \mu \) possibly different from \( \mathbf{0} \), then (1) is modified to \((\mathbf{X}_2 - \mu) = \mathbf{B}(\mathbf{X}_1 - \mu) + \mathbf{U}_2 \) or (1) holds with \( \mathbf{U}_2 \) having the distribution \( \mathcal{N}(\mathbf{v}, \mathbf{\Sigma}) \), where \( \mathbf{v} = (\mathbf{I} - \mathbf{B}) \mathbf{\mu} \).

**REFERENCES**