On Some Contributions of Charles Stein to Applications of Invariance in Statistics

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Three Selected Contributions: This presentation is focused mainly on the following triptych based primarily on the insights of Dr. Stein.

- The Hunt-Stein (HS) condition (1945-1946) and its aftermath.

- "Gl_p is not amenable" (1956) and the implications.

HS and its Aftermath: The original Hunt-Stein work of 1945 concerned invariant testing problems and the use of the invariance in establishing that certain tests were most stringent (see Lehmann(1950) for discussion). This work quickly (??) lead to:

- The formulation of general invariant decision problems (Peisakoff(1950) —Princeton Ph.D. thesis, unpublished; Kudo(1955), and Stein(1956)).

- The HS condition: On a topological group $G$, assume there is a sequence of probability measures $\nu_n$ such that

\[|\nu_n(B) - \nu_n(Bg)| \rightarrow 0\]

for each Borel $B$ and each $g \in G$. This is often called the HS condition (see Bondar and Milnes(1981)).
The Invariant Minimax Theorem

- For an invariant decision problem, when HS holds, the Invariant Minimax Theorem (IMT) holds — that is, the minimax risk is equal to the invariant minimax risk.
Following the Hunt-Stein work: Early (1950-1957) works that flowed from the Hunt-Stein contributions include:

- Lehmann (1950) - invariant testing problems and a discussion of the HS work

- Peisakoff (1950) - a version of the IMT and an example where IMT does not hold (free group on two generators).

- Kudo (1955) - a version of IMT and a proof that $G_T$ (the lower triangular group) satisfies HS.

- Stein (1956, TR6, Stanford) - some examples from multivariate analysis

- Kiefer (1957) - IMT holds when HS holds
**$GL_p$ is not amenable and its implications:** Many problems in multivariate analysis are invariant under $GL_p$ (or groups directly related $GL_p$). In such problems, inferences based on the likelihood function will be invariant. Thus trouble for $GL_p$ invariant rules will automatically imply trouble for many classical procedures.

- Stein (1955, 1956) argued via examples that there seemed to be problems with $GL_p$ invariant procedures — at least from the decision theoretic point of view. Stein had already shown that the MLE of the mean vector was inadmissible when dimension was at least 3 and the loss was quadratic (that is, a best invariant procedure was inadmissible).
• A main result in Stein(1956) was that no constant multiple of the sample covariance matrix can be minimax for estimating $\Sigma$ when using Stein’s loss (and most likely many other $G_{lp}$ invariant loss functions).

• The idea of the proof is to look at $G_T$ (lower triangulars with positive diagonals). Kudo(1955) had proved that this group satisfies HS so the IMT holds for $G_T$. 
• In Stein(1956) a unique best $G_T$-invariant estimator of $\Sigma$ is derived. This estimator is not a multiple of $S$, but from Kudo, it is minimax. But every $Gl_p$ invariant estimator is a multiple of $S$, and hence cannot be minimax (these are uniformly inadmissible). Thus $Gl_p$ cannot satisfy HS (it cannot be amenable).

• The non-amenability is of consequence not only for the estimation of $\Sigma$, but for a number of inferential problems in multivariate analysis. One striking example concerns prediction where the ”inferences” are predictive distributions. A number of $Gl_p$ invariant predictive distributions have been proposed, but all such inferences are ”incoherent” (de Finetti) and ”strongly inconsistent” (Stone). Alternatives have been proposed, but these are ”coordinate choice” dependent as is the best $G_T$-invariant estimator.
Using the Right Haar(RH) Prior: In 1963, Stein argued that the RH prior will produce a posterior (the RH posterior) that produces exact probability matching, assuming that the group acts transitively on the parameter space. As a student, I learned from Dr. Stein that using a RH prior will produce a best invariant decision rule via the formal Bayes method (assuming transitivity).

- The Stein result extends the Pitman paper. By probability matching we mean that one uses the RH posterior to produce a HPD region of constant (posterior) probability. This region then has the same constant frequency probability of confidence coverage. A subtle but important point is that the argument does not depend on Fisherian-pivoting and thus yields wider application.
• In an invariant decision problem with a transitive group action, all invariant rules have constant risk. Thus a "best" invariant rule should exist. Typically the formal Bayes method of minimizing the posterior risk relative to the RH posterior will produce a best invariant rule (I learned this from Dr. Stein about 1966, but the provenance is not known to me. Zidek(1969) says the above assertion is "well known", but does not give a reference.)

• Using the RH prior may lead to a reasonable posterior, but the posterior may be incoherent (de Finetti) for some groups — eg: the group $Gl_p$. However if the group satisfies HS, then the posterior will not be incoherent.