# NOTES

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## Fair Dice*

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**1. Introduction.** Dice are usually cubes of a homogeneous material. Symmetry suggests that a homogeneous cube has the same chance of landing on each of its six faces after a vigorous roll, so it is said to be fair. Similarly the four other regular solids—the tetrahedron, octahedron, dodecahedron and icosahedron—are fair. Are there any other fair polyhedra?

To answer this question we must first define what we mean by fair. We shall say that a convex polyhedron is fair by symmetry if and only if it is symmetric with respect to all its faces. This means that any face can be transformed into any other face by a rotation, a reflection, or a combined rotation and reflection, which takes the polyhedron into itself. The collection of all these transformations of a given polyhedron is called its symmetry group. The fact that some transformation in the group takes any given face into any other given face is expressed by saying that the group acts transitively on the faces. Thus we can say that a convex polyhedron is fair by symmetry if and only if its symmetry group acts transitively on its faces.

In the next section we shall determine all such polyhedra. Then in the final section we shall show that there are other polyhedra which are fair, but not fair by symmetry.

**2. Polar reciprocals.** All the symmetry transformations of a convex polyhedron leave invariant its center of gravity, which is a point inside the polyhedron. We form the dual or polar reciprocal polyhedron of the original polyhedron with respect to its center. This is done by passing a plane through each vertex of the original polyhedron, perpendicular to the line from the center to the vertex. These planes form the faces of the polar reciprocal polyhedron, and each of its vertices is called a pole of one face of the original polyhedron.

Because the symmetry group acts transitively on the faces of the original polyhedron, it also acts transitively on their poles, which are the vertices of the polar reciprocal polyhedron. Therefore the polar reciprocal polyhedron is symmetric with respect to its vertices.

The symmetry groups of all polyhedra symmetric with respect to their vertices have been determined [1]. Furthermore for each such group there is one polyhedron

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which has regular polygons for faces. These particular polyhedra are the well-known semiregular solids. They comprise thirteen individuals, the Archimedean solids, and the two infinite classes of prisms and anti-prisms, which were recognized as semiregular by Kepler. The regular or Platonic solids are also semiregular.

Now we use the fact that the polar reciprocal of the polar reciprocal of a polyhedron is similar to the original polyhedron. Therefore we have obtained the following result: The polyhedra which are fair by symmetry are duals of the polyhedra symmetric with respect to their vertices. Each symmetry group of a fair polyhedron is represented by a regular solid or the dual of a semiregular solid. Thus in addition to the five regular solids there are thirteen individual polyhedra and two infinite classes among the fair polyhedra.

Lists of the regular and semiregular solids are given on pages 272 and 277 of reference 1, together with some of their properties. From the list on page 272 we see that every semiregular solid has an even number of vertices, so its polar reciprocal has an even number of faces. Therefore every polyhedron fair by symmetry has an even number of faces. Drawings of the semiregular solids are shown on page 280, and they are determined on pages 269–277 and again on pages 279–286. Drawings of the polar reciprocals of the semiregular solids are shown on pages 34 and 35 of Pearce [**2**], and photographs of models of them are shown on page 54 of Holden [**3**].

Each of the semiregular solids belongs to a class of polyhedra with the same symmetry group, all of which are symmetric with respect to their vertices. The dual of any one of them is fair by symmetry, but in general it will be less symmetric than the dual of the semiregular solid. For example, suppose that each equilateral triangular face of a semiregular antiprism is replaced by a given isoceles triangle which is not equilateral. The resulting solid is symmetric with respect to its vertices and has the same symmetry group as the semiregular antiprism. Its dual is a fair die with quadrilateral faces which are not symmetric about either of their diagonals. On the other hand, suppose that the square faces of a semiregular prism are replaced by nonsquare rectangles. The new polyhedron is symmetric with respect to its vertices and has the same symmetry group as the semiregular prism. Its dual is again a dipyramid which is fair by symmetry. It differs from the dual of the semiregular prism only in the ratio of the lengths of the sides of a face.

Grünbaum and Shephard [**4**] give a complete classification of the convex polyhedra with symmetries acting transitively on their faces, which they call "isohedra." They are classified by combinatorial isomorphism type and by symmetry group. There are some isohedra for which the positive rotations alone (i.e. those which preserve orientation, so they do not include reflections) do not act transitively on the faces. Earlier Grünbaum [**5**] showed that every isohedron has an even number of faces.

**3. Other fair polyhedra.** There are other fair polyhedra which are not symmetric. To show this we consider, for example, the dual of the $n$-prism, which is a dipyramid with $2n$ identical triangular faces. We cut off its two tips with two planes parallel to the base and equidistant from it. When the cuts are near the tips, the solid has a very small probability of landing on either of the two tiny new faces. However when the cuts are near the base, it has a very high probability of landing on one of them. Therefore by continuity there must be cuts for which the two new faces and the $2n$ old faces have equal probabilities. The locations of those cuts, which depend upon the mechanical properties of the die and the table, could be found by experiment or by a difficult mechanical analysis along the lines of our previous study of coin tossing [**6**]. Similar constructions can be carried out starting with other dice which are fair by symmetry, or which are obtained by this construction. We say these dice are fair by continuity.

As an example, let us consider an infinite prism with a regular $n$-gon as its cross section. Let us cut it with two planes a distance $L$ apart perpendicular to its generators, to produce a polyhedron with $n + 2$ faces. For $L$ large this solid has very low probability of landing on either of its two ends, whereas for very small $L$ it has a high probability of landing on one of them. Therefore by continuity there is some value of $L$ for which it has the same probability of landing on any one of its $n + 2$ faces. When $n$ is odd this yields a fair die with an odd number of faces.

The problem of characterizing all fair dice, not just those which are fair by symmetry or by continuity, is still unsolved.

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# A Remark on Euclid's Proof of the Infinitude of Primes

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IN MEMORY OF NIKHIL BANERJEE

Every student of elementary number theory knows Euclid's proof that there are infinitely many prime numbers: for if $p_1, \dots, p_n$ are the first $n$ primes and $N$ is their product, then *every* prime dividing $(N + 1)$ is larger than $p_n$. If, however, we let $N_i^{(n)}$ be the product of the first $n$ primes with the prime $p_i$ $(1 \leqslant i \leqslant n)$ excluded, it does *not* follow that *every* prime divisor of $(N_i^{(n)} + 1)$ is larger than $p_n$, as the prime $p_i$ is now a possible divisor, for example:

(a) $N_2^{(5)} + 1 = 2 \cdot 5 \cdot 7 \cdot 11 + 1 = 3 \cdot 257$, is divisible by the primes 3 and 257,
$N_4^{(6)} + 1 = 2 \cdot 3 \cdot 5 \cdot 11 \cdot 13 + 1 = 7 \cdot 613$, divisible by the primes 7 and 613.

(b) $N_1^{(3)} + 1 = 3 \cdot 5 + 1 = 2^4$, is divisible only by the prime 2,
$N_1^{(2)} + 1 = 3 + 1 = 2^2$, also divisible only by the prime 2,
$N_2^{(2)} + 1 = 2 + 1 = 3$.