Stats 318: Lecture 2

Agenda:

▶ Epidemiology Example
▶ MC and Random Recurrent Sequences
▶ Stationary Distributions
▶ Reversibility
▶ Existence of Stationary Distributions
Random Recurrent Sequence

\[ X_{t+1} = f(X_t, \theta_{t+1}) \]

(i) \( \theta_1, \theta_2, \ldots, \) iid RVs taking values in \( \Theta \)

(ii) Map \( f : \Theta \times X \to X \) is measurable

(iii) \( X_0 \) is a RV (possibly deterministic) independent of \( \{\theta_t\}_{t \geq 1} \)

If \( f \) does not depend on \( \theta \), recurrence is deterministic

- Dynamical Systems Theory if \( X_0 \) deterministic
- Ergodic Theory if \( X_0 \) random
Canonical Representation

Claim
(i) A random dynamical system is a Markov chain

(ii) Conversely, every Markov chain with transition matrix $P$ can be expressed as a random dynamical system

Proof
(i)

\[
P(X_{t+1} = x_{t+1} | X_{0:t} = x_{0:t}) = \mathbb{P}(f(x_t, \theta_{t+1}) = x_{t+1} | X_{0:t} = x_{0:t})
\]

\[
= \mathbb{P}(f(\theta_{t+1}, x_t) = x_{t+1}) = P(x_t, x_{t+1})
\]
Proof

(ii) Finite/countable state space: e.g. $\mathcal{X} = \{1, \ldots, n\}$

For $i, j \in \mathcal{X}$ set $\tau(i, j) = p_{i1} + \ldots + p_{ij}$  \quad $\tau(i, 0) = 0$

Now $\theta \sim U[0, 1]$ and define $f : [0, 1] \times \mathcal{X} \rightarrow \mathcal{X}$ by

$$f(\theta, i) = j \quad \text{if} \quad \tau(i, j - 1) \leq \theta < \tau(i, j)$$

$$P(X_{t+1} = j | X_t = i) = p_{i,j} = P(\theta_{t+1} \in [\tau(i, j - 1), \tau(i, j))$$

\[\text{Diagram:} \quad \text{State transitions and probabilities}\]
Kolmogorov Chapman Equations

- Initial distribution: $X_0 \sim \pi^{(0)}$

- Vector matrix product
  \[
  (\pi P)(x) = \sum_y \pi(y)P(y, x)
  \]

- Interpretation
  \[
  X_0 \sim \pi^{(0)} \implies X_1 \sim \pi^{(1)} = \pi^{(0)}P
  \]

- $P^n$: $n$-th power of matrix $P$
  \[
  P^{n+1}(y, x) = (P \cdot P^n)(y, x) = \sum_z P(y, z)P^n(z, x)
  \]
Kolmogorov Chapman Equations

<table>
<thead>
<tr>
<th>Theorem</th>
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<tbody>
<tr>
<td>( {X_t} ) is a MC with state space ( \mathcal{X} )</td>
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<td>(i) ( X_t \sim \pi^{(t)} ) obeying recurrence relation</td>
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<td>[ \pi^{(t)} = \pi^{(t-1)} P = \pi^{(0)} P^t ]</td>
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<td>(ii) for all ( x, y \in \mathcal{X} )</td>
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<tr>
<td>[ P(X_t = x</td>
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<td>(iii) For all bounded functions ( h : \mathcal{X} \to \mathbb{R} )</td>
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<td>[ \mathbb{E}[h(X_t)</td>
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Proof

(i) \( \pi^{(t+1)}(x) = P(X_{t+1} = m) \)

\[
= \sum_y P(X_{t+1} = m | X_t = y) P(X_t = y)
\]

\[
= \sum_y \pi^{(t)}(y) P(y, x)
\]

\[
= (\pi^{(t)} P)(x)
\]

(ii) Apply (i) with \( \pi^{(0)} = \delta(y) \)

(iii) \( \mathbb{E}[h(X_t)] = \sum_x h(x) [\pi^{(0)} P^{(t)}](x) \)

with \( \pi^{(0)} = \delta(y) \), we have

\( \mathbb{E}[h(X_t) | X_0 = y] = \sum_x h(x) P^t(y, x) = (P^t h)(x) \)
Example

- Wish to transmit an \( n \)-bit signal through a channel
- Each bit is transmitted with an error probability
  \[
  0 \rightarrow 1 \quad \text{w.p. } a \quad 0 < a, b < 1
  \]
  \[
  1 \rightarrow 0 \quad \text{w.p. } b
  \]
- Transmitted signal after \( t \) steps is \( X_t \)
  \[
  \{0, 1, 0, 0, \ldots, 0\} \rightarrow \{0, 1, 1, 0, \ldots, 0\} \rightarrow \ldots \rightarrow X_0 \xrightarrow{X_1} \ldots \xrightarrow{X_t}
  \]
- Relays operate independently of each other
- Bit errors are independent

Q: Critical number of relays beyond which probability of receiving an incorrect message is above a tolerance level \( \varepsilon \)?
Suppose first $n = 1$
Sequence $X_t$ has transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Put $g_t = P(X_t = 0)$. From theorem,

$$g_{t+1} = (1-a)g_t + b(1-g_t)$$

Fixed point $g = \frac{b}{a+b}$

$$g_t - g = (1-a-b)^t(g_0 - g)$$

Probability received message is correct:

$$r_t(0) = \frac{b}{a+b} + \frac{a}{a+b}(1-a-b)^t \quad X_0 = 0$$

$$r_t(1) = \frac{a}{a+b} + \frac{b}{a+b}(1-a-b)^t \quad X_0 = 1$$
Length message $n \geq 1$

$X_t = (X_t^1, \ldots, X_t^n)$ where variables are independent and have the same distribution for the same initial value of the bit

$$r_t = \prod_{i=1}^{n} r_t(X^i_0) \geq [\alpha + (1 - \alpha)(1 - a - b)^t]^n \quad \alpha = \min(g, 1 - g)$$

$RHS$

choose $t$ s.t. $1 - RHS \leq \varepsilon$
Stationary Distributions/Equilibrium Measures

**Definition**

Let $\{X_t\}_{t \geq 0}$ be a Markov Chain with transition matrix $P$. A probability distribution $\pi$ is said to be invariant or stationary if

$$\pi P = P$$

Interpretation: If initial distribution $\pi^{(0)} = \pi$, then

$$\pi^{(1)} = \pi P = \pi$$

and by integrating $\pi^{(t)} = \pi \quad \forall \ t \geq 1$

From statistical viewpoint, system is in equilibrium

Other name: Equilibrium Distribution or Measure