1. Let $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ be orthogonal matrices. Show that for any $A \in \mathbb{R}^{m \times n}$, $\|UAV\|_F = \|A\|_F$ and $\|UAV\|_2 = \|A\|_2$. Is it true that we also always have $\|UAV\|_1 = \|A\|_1$? Why or why not?

2. Let $A \in \mathbb{R}^{m \times n}$ and recall that the $p$-norm of $A$ is defined as
   \[ \|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p. \]

   (a) Show that
   \[ \|A\|_1 = \max_{1 \leq j \leq n} c_j, \quad c_j = \sum_{i=1}^{m} |a_{ij}|. \]

   (b) Show that
   \[ \|A\|_\infty = \max_{1 \leq i \leq m} r_i, \quad r_i = \sum_{j=1}^{n} |a_{ij}|. \]

3. Very simple linear regression. Consider the three points
   \[ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -2 \end{bmatrix}. \]

   What is the equation of the line that best fits those three points, in the sense of least squares?

4. The least-squares solution minimizes $\|b - Ax\|^2$. In some applications, we would like to favor ‘low-energy’ solutions and minimize instead
   \[ \|b - Ax\|^2 + \lambda \|x\|^2, \quad (1) \]

   where $\lambda$ is a positive parameter chosen by the scientist/engineer.

   (a) Find the solution to this problem. [Hint: The SVD of $A$ might be quite handy here.]

   (b) Suppose you have a routine that returns the SVD of any matrix $A$; that is, it returns $U$, $\Sigma$ and $V$. Can you use it to compute the solution? If yes, explain how.

   (c) Suppose you have a routine that returns the solution to any least-squares problem; that is, it takes as input an arbitrary pair $A$, $b$ and returns an $x$ minimizing $\|b - Ax\|^2$. Can you use it to compute the solution? If yes, explain how.