Stats 318: Lecture

Agenda: Markov Chain Monte Carlo

- The Propp Wilson algorithm
- Exact sampling of the Ising model
Ising Model

- $n$ by $n$ ‘spin’ array
- $I_v = \pm 1, \, v = (v_1, v_2), \, 1 \leq v_1, v_2 \leq n$
- Energy

$$E(I) = -\frac{1}{2} \sum_{v \sim v'} I_v I_{v'}$$

where the $\text{sym } v \sim v'$ symbolizes all pairs of nearest neighbours on the lattice.

- Boltzmann distribution

$$\pi(I) = \frac{e^{-\beta E(I)}}{\sum_{\text{all states } I'} e^{-\beta E(I')}}$$
Recall Gibbs sampler for the Ising model

\[
T = 1; \quad \text{beta} = 1/T;
\]

% Initialize the chain
X = ones(10,10);

% Number of steps
N = 10000;

for n = 1:N,
    % Pick a vertex uniformly at random
    v = randsample(1:10,2,true);

    % Sample from the conditional distribution
    G = sum(neighbors(v,X));
    if (rand(1) < 1/(1+exp(-beta*G)))
        X(v(1), v(2)) = 1;
    else
        X(v(1), v(2)) = -1;
    end
end
Propp Wilson algorithm

% Coupling from the past: Exact sampling of the Ising Model by Gibbs sampling
T = 1; beta = 1/T;

% Initialization
Xmax = ones(10,10); Xmin = -Xmax;
k = 1;
V = reshape(randsample(1:10,2^k,true),2,1);
U = rand(1);

while (~CompareArrays(Xmin,Xmax))

    % Pick 2^(k-1) vertices uniformly at random
    V = [reshape(randsample(1:10,2^k,true),2,2^(k-1)) V];
    % Pick 2^k uniform variables (to use for changing state)
    U = [rand(1,2^(k-1)) U];
Xmax = ones(10,10); Xmin = -Xmax;
% Move both chains from time \(-2^k\) to time 0
for n = 1:2^k,
    v = V(:,n)'; u = U(n);
    Gmin = sum(neighbors(v,Xmin)); Gmax = sum(neighbors(v,Xmax));

    if (u < 1/(1+exp(-beta*Gmin)))
        Xmin(v(1), v(2)) = 1;
    else
        Xmin(v(1), v(2)) = -1;
    end

    if (u < 1/(1+exp(-beta*Gmax)))
        Xmax(v(1), v(2)) = 1;
    else
        Xmax(v(1), v(2)) = -1;
    end
end

k = k+1;
end
function identical = CompareArrays(X,Y)

ncoordinates = prod(size(X));
nequal = sum(sum(X==Y));
identical = (nequal == ncoordinates);
Magnetization

Magnetization is: $M = \sum_v I_v$
After how many steps?

For $T = 3$ and startup times of the form $-2^k$, $k = 1, 2, 3, \ldots$, convergence occurred for

- $k = 9$ in 1 instance
- $k = 10$ in 51 instances
- $k = 11$ in 49 instances

For $T = 1$ (very cold temperature), convergence is of course slower; takes around $2^{16}$ steps.