1. We would like to understand the finite sample properties of the higher criticism (HC). Throughout this problem, we will work with $n = 10^6$ independent observations $X_i \sim N(\mu_i, 1)$. Under the global null $H_0$, all the $\mu_i$’s are equal to zero while under $H_1$, $n^{1-\beta}$ of the $\mu_i$’s are equal to $\sqrt{2r(\beta) \log n}$, where

$$r(\beta) = 1.2\rho^*(\beta) + 0.1, \quad \rho^*(\beta) = \begin{cases} \beta - 1/2, & \frac{1}{2} < \beta \leq \frac{3}{4}; \\ (1 - \sqrt{1-\beta})^2, & \frac{3}{4} < \beta \leq 1; \end{cases}$$

$\rho^*(\beta)$ is the critical threshold above which the HC statistic should theoretically have full asymptotic power. Throughout, we work with values of $\beta \in \{0.50, 0.55, 0.60, \ldots, 1\}$, and a significance level set to $\alpha = 0.05$.

(a) Use Monte-Carlo simulations to compute an approximate threshold for the higher criticism statistic

$$\max_{1 \leq i \leq n/2} \sqrt{n} \cdot \frac{i/n - p(i)}{\sqrt{p(i)(1 - p(i))}}$$

(here, $p(i)$ are the sorted p-values for testing $\mu_i = 0$ vs. $\mu_i > 0$). For what values of the index $i$ is this maximum typically achieved? Can you explain why?

(b) In the setup described above, compute the power of the HC statistic through simulations. Is the power close to one? Has the asymptotics kicked in?

(c) Consider an alternative to HC, where one would standardize things differently: namely, define the log-likelihood transformation

$$\frac{1}{n} \log(LR_n(t)) := \begin{cases} F_n(t) \log \left( \frac{F_n(t)}{t} \right) + (1 - F_n(t)) \log \left( \frac{1 - F_n(t)}{1 - t} \right), & 0 < t < F_n(t), \\ 0, & \text{otherwise} \end{cases}$$

This is the one sided log-likelihood ratio for testing whether the parameter of the binomial $nF_n(t)$ equals $t$ vs. whether it is larger than $t$. We consider the Berk-Jones statistic rejecting when

$$BJ_n = \max_{1 \leq i \leq n/2} \log(LR_n(p(i)))$$

is large. Why does this make sense? Explain why this may be better than the HC, at least in some regimes.

(d) Repeat (b) with the Berk-Jones statistic. What do you see? Comment on differences and offer an explanation as to why they occur.

(e) What would you recommend using in practice? Do you see a way of combining the BJ and the HC statistics?

2. Simes test is actually conservative under a form of positive dependence. This is not a simple result and in this problem, we will assume that the test statistics under the global null follow a multivariate Gaussian distribution $N(\mu, \Sigma)$ with known covariance $\Sigma$. To make life simpler, we will assume that the variances are all equal to 1, i.e. $\Sigma_{i,i} = 1$. Say we are interesting in $H_i : \mu_i \leq 0$.

(a) Propose and carry out a series of simulations which show that under positive correlations, Simes test is conservative.

(b) Through simulations, show that under strong correlations, Simes test may have much more power than the Bonferroni method.