Supplemental Materials for:
“The Likelihood Ratio Test in High-Dimensional Logistic Regression Is Asymptotically a Rescaled Chi-Square”

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Abstract
This document presents the proof of Lemma 6(ii) given in the paper [1]: “The Likelihood Ratio Test in High-Dimensional Logistic Regression Is Asymptotically a Rescaled Chi-Square”.

1 Proof of Lemma 6(ii)
We shall prove that $V(\tau^2) < \tau^2$ whenever $\tau^2$ is sufficiently large. Before proceeding, we recall from the main text and [2, Proposition 6.4] that

\[ V(\tau^2) := \frac{1}{\kappa} \mathbb{E} \left[ \Psi^2(\tau Z; b(\tau)) \right] = \frac{1}{\kappa} \mathbb{E} \left[ \left( b(\tau) \rho' \left( \text{prox}_{\theta b(\tau)} (\tau Z) \right) \right)^2 \right], \]  

where $b(\tau)$ obeys

\[ \kappa = \mathbb{E} \left[ \Psi' (\tau Z; b(\tau)) \right] = 1 - \mathbb{E} \left[ \frac{1}{1 + b(\tau) \rho'' \left( \text{prox}_{\theta b(\tau)} (\tau Z) \right)} \right]. \]  

In what follows, we study the logistic and probit models separately.

1.1 The logistic case
Consider the bivariate functions

\[ h(t, \tau) := \mathbb{E} \left[ \frac{1}{1 + b \rho'' (\text{prox}_{\theta b} (\tau Z))} \right], \]  

\[ w(t, \tau) = \mathbb{E} \left[ (\rho' (\text{prox}_{\theta b} (\tau Z)))^2 \right], \]

which plays a central role in [1] and [2]. In the sequel, we will first analyze these two functions for any $b$ obeying

\[ b = c_0 \tau \]

for some constant $c_0 > 0$. The result is this:

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Lemma 1. For any constant $c_0 > 0$, one has
\[
\lim_{\tau \to \infty} h(c_0 \tau, \tau) = \mathbb{P} \{ Z < 0 \text{ or } Z > c_0 \};
\]
\[
\lim_{\tau \to \infty} w(c_0 \tau, \tau) = \mathbb{P} \{ Z > c_0 \} + \frac{1}{c_0^2} \mathbb{E} \left[ Z^2 \mathbb{1}_{\{0 < Z < c_0\}} \right].
\]

Recall that $0 < \kappa < 1/2$. One can easily find two constants $c_0 > \hat{c}_0 > 0$ such that
\[
\mathbb{P} \{ Z < 0 \text{ or } Z > c_0 \} < 1 - \kappa < \mathbb{P} \{ Z < 0 \text{ or } Z > \hat{c}_0 \}.
\]
In view of Lemma 1, for any sufficiently large $\tau > 0$ one has
\[
h(c_0 \tau, \tau) < 1 - \kappa = h(b(\tau), \tau) < h(\hat{c}_0 \tau, \tau).
\]
According to Lemma 5, $h(b, \tau)$ is a monotonic function in $b$ for any given $\tau > 0$, thus indicating that $b(\tau) \in [\hat{c}_0 \tau, c_0 \tau]$; that said, $b(\tau)$ scales linearly in $\tau$ as $\tau \to \infty$. Furthermore, since $b(\tau)$ is the solution to $h(b, \tau) = 1 - \kappa$, one has
\[
\lim_{\tau \to \infty} \mathbb{P} \left\{ Z < 0 \text{ or } Z > \frac{b(\tau)}{\tau} \right\} = 1 - \kappa,
\]
which leads to the closed-form expression
\[
\lim_{\tau \to \infty} \frac{b(\tau)}{\tau} = \Phi^{-1}(\kappa + 0.5).
\]

We are now ready to characterize the variance map. Note that when $\tau$ is sufficiently large,
\[
\frac{\mathcal{V}(\tau^2)}{\tau^2} = \frac{b^2(\tau)}{\tau^2} \cdot \frac{\mathbb{E} \left[ h'(\text{prox}_{b(\tau)}^\tau(\tau Z)) \right]^2}{1 - \mathbb{E} \left[ 1 + b(\tau) h'(\text{prox}_{b(\tau)}^\tau(\tau Z)) \right]}
\]
\[
= (1 + o(1)) \frac{b^2(\tau)}{\tau^2} \left\{ \mathbb{P} \left\{ Z > \frac{b(\tau)}{\tau} \right\} + \frac{\tau^2}{\mathbb{E} \{ b(\tau) \}} \mathbb{E} \left[ Z^2 \mathbb{1}_{\{0 < Z < \frac{b(\tau)}{\tau}\}} \right] \right\}
\]
\[
= (1 + o(1)) \frac{\tau^2 \mathbb{P} \{ Z > x \} + \mathbb{E} \left[ Z^2 \mathbb{1}_{\{0 < Z < x\}} \right]}{\mathbb{P} \{ 0 < Z < x \}} \bigg|_{x = \frac{b(\tau)}{\tau}}.
\]

This together with the expression of $\frac{b(\tau)}{\tau}$ in [5] gives
\[
\lim_{\tau \to \infty} \frac{\mathcal{V}(\tau^2)}{\tau^2} = \frac{\tau^2 \mathbb{P} \{ Z > x \} + \mathbb{E} \left[ Z^2 \mathbb{1}_{\{0 < Z < x\}} \right]}{\mathbb{P} \{ 0 < Z < x \}} \bigg|_{x = \Phi^{-1}(\kappa + 0.5)}.
\]

In order to prove that $\mathcal{V}(\tau^2) \leq \tau^2$ for large $\tau$, it suffices to show that the function
\[
g(x) := x^2 \mathbb{P} \{ Z > x \} + \mathbb{E} \left[ Z^2 \mathbb{1}_{\{0 < Z < x\}} \right] - \mathbb{P} \{ 0 < Z < x \}
\]
obeys $g(x) < 0$ for all $x > 0$. To this end, some algebra gives
\[
g(x) = x^2 \int_x^\infty \phi(z) \, dz + \int_0^x z^2 \phi(z) \, dz - \int_0^x \phi(z) \, dz
\]
\[
= x^2 \int_x^\infty \phi(z) \, dz - x \phi(x) \bigg|_0^x + \int_0^x \phi(z) \, dz - \int_0^x \phi(z) \, dz
\]
\[
= x \left( x \int_x^\infty \phi(z) \, dz - \phi(x) \right) < 0,
\]

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where (10) comes from integration by parts, and the last inequality follows from \( \int_{x}^{\infty} \phi(z) \, dz < \frac{1}{2} \phi(x) \). This establishes that \( \mathcal{V}(\tau^2) \leq \tau^2 \) for any sufficiently large \( \tau > 0 \).

Finally, we prove Lemma 1.

**Proof of Lemma 1** Take \( \epsilon > 0 \) to be an arbitrarily small constant. We study \( \frac{1}{1 + b \rho''(\prox b \phi \tau Z)} \) and \( (\rho' (\prox b \phi \tau Z))^2 \) in three separate cases.

- **Case 1:** \( Z \leq -\epsilon \). Recall that \( \prox b \phi \tau Z \) is the solution to

\[
\frac{b \, e^t}{1 + e^t} + t = \tau Z, \tag{11}
\]

which implies that

\[
\prox b \phi \tau Z = \tau Z - b \frac{e^t}{1 + e^t} \bigg|_{t = \prox b \phi \tau Z} < \tau Z \leq -\epsilon \tau. \tag{12}
\]

When \( \tau \to \infty \), this yields

\[
0 \leq b \rho''(\prox b \phi \tau Z) = b \frac{e^t}{(1 + e^t)^2} \bigg|_{t = \prox b \phi \tau Z} \leq b e^t \bigg|_{t = \prox b \phi \tau Z} \leq c_0 \tau e^{-\epsilon \tau} \to 0,
\]

or equivalently,

\[
1 - \frac{1}{1 + b \rho''(\prox b \phi \tau Z)} \to 0 \quad \text{as} \quad \tau \to \infty.
\]

Similarly, one can derive

\[
(\rho' (\prox b \phi \tau Z))^2 = \frac{e^{2t}}{(1 + e^t)^2} \bigg|_{t = \prox b \phi \tau Z} \leq e^{2\prox b \phi \tau Z} \leq e^{-\epsilon \tau} \to 0,
\]

where (a) follows from (12).

- **Case 2:** \( Z \geq \frac{b}{\tau} + \epsilon \). In this case, it holds that

\[
\prox b \phi \tau Z = \tau Z - b \frac{e^t}{1 + e^t} \bigg|_{t = \prox b \phi \tau Z} > \tau \left( \frac{b}{\tau} + \epsilon \right) - b = \epsilon \tau.
\]

Applying a similar argument as in the previous case, we see that as \( \tau \to \infty \),

\[
1 - \frac{1}{1 + b \rho''(\prox b \phi \tau Z)} \to 0 \quad \text{and} \quad (\rho' (\prox b \phi \tau Z))^2 \to 1.
\]

- **Case 3:** \( \epsilon < Z < \frac{b}{\tau} - \epsilon \). We can first rule out the possibility of \( |\prox b \phi \tau Z| \gtrsim \tau \). In fact, if \( |\prox b \phi \tau Z| \gtrsim \tau \) and \( \prox b \phi \tau Z \geq 0 \), then

\[
\frac{b \, e^t}{1 + e^t} \bigg|_{t = \prox b \phi \tau Z} + \prox b \phi \tau Z \geq b \frac{e^t}{1 + e^t} \bigg|_{t = \prox b \phi \tau Z} = b - \frac{b}{1 + e^{\prox b \phi \tau Z}} \geq b - \frac{b_0}{e^{b_0(\tau)}} > b - \epsilon \tau > \tau Z,
\]

where (b) follows from the assumptions \( b_0 = c_0 \) and \( |\prox b \phi \tau Z| \gtrsim \tau \), and (c) holds when \( \tau \) is sufficiently large. This violates the identity \( 11 \). Similarly, if \( |\prox b \phi \tau Z| \gtrsim \tau \) and \( \prox b \phi \tau Z < 0 \), then

\[
\frac{b \, e^t}{1 + e^t} \bigg|_{t = \prox b \phi \tau Z} + \prox b \phi \tau Z < b \frac{e^{\prox b \phi \tau Z}}{1 + e^{\prox b \phi \tau Z}} = c_0 \tau \frac{e^{-|\prox b \phi \tau Z|}}{1 + e^{-|\prox b \phi \tau Z|}} \quad \text{(d)}
\]

\[
\quad \quad \quad \quad < \epsilon \tau \leq \tau Z,
\]

where \( c_0 \) is an arbitrarily small constant.
where (d) follows when \( \tau \) is sufficiently large. This inequality contradicts (11) as well. As a result, we reach
\[
\left| \text{prox}_{b\rho}(\tau Z) \right| = o(1) \text{ in this case, which combined with (11) gives}
\]
\[
b \frac{e^t}{1 + e^t} \bigg| \tau = \text{prox}_{b\rho}(\tau Z) = (1 + o(1)) \tau Z.
\]
(13)

Additionally, (13) leads to
\[
\frac{1}{1 + e^t} \bigg| \tau = \text{prox}_{b\rho}(\tau Z) = (1 + o(1)) \left(1 - \frac{\tau Z}{b}\right),
\]
(14)

which is bounded away from 0 in this case. Taken together, (13) and (14) yield
\[
\frac{1}{1 + b \rho''(\text{prox}_{b\rho}(\tau Z))} = \frac{1}{1 + b \frac{e^t}{(1 + e^t)^2} \bigg| \tau = \text{prox}_{b\rho}(\tau Z)} = \frac{1}{1 + (1 + o(1)) \tau Z (1 - \frac{\tau Z}{b})} \to 0
\]
and
\[
(\rho'(\text{prox}_{b\rho}(\tau Z)))^2 = \left( \frac{e^t}{1 + e^t} \right)^2 \bigg| \tau = \text{prox}_{b\rho}(\tau Z) = (1 + o(1)) \frac{\tau z^2}{b^2}.
\]

Putting the above cases together and applying dominated convergence gives
\[
\lim_{\tau \to \infty} \left\{ \mathbb{E} \left[ \frac{1}{1 + b \rho''(\text{prox}_{b\rho}(\tau Z))} \right] - \mathbb{E} \left[ \frac{1}{1 + b \rho''(\text{prox}_{b\rho}(\tau Z))} \right] \mathbb{1}_{\{|Z| \leq \varepsilon \text{ or } |Z - b/\tau| \leq \varepsilon\}} \right\}
\]
\[
= \lim_{\tau \to \infty} \left\{ \mathbb{E} \left[ \mathbb{1}_{\{Z < -\varepsilon\}} \right] + \mathbb{E} \left[ \mathbb{1}_{\{Z > b/\tau - \varepsilon\}} \right] \right\} = \lim_{\tau \to \infty} \mathbb{P} \left\{ Z < -\varepsilon \text{ or } Z > \frac{b}{\tau} + \varepsilon \right\}
\]
when \( b = c_0 \tau \) for some constant \( c_0 > 0 \). Recognizing that
\[
\mathbb{E} \left[ \frac{1}{1 + b \rho''(\text{prox}_{b\rho}(\tau Z))} \right] \mathbb{1}_{\{|Z| \leq \varepsilon \text{ or } |Z - b/\tau| \leq \varepsilon\}} \leq \mathbb{E} \left[ \mathbb{1}_{\{|Z| \leq \varepsilon \text{ or } |Z - b/\tau| \leq \varepsilon\}} \right] \leq 4\varepsilon
\]
and
\[
\mathbb{P} \left\{ -\varepsilon \leq Z \leq 0 \text{ or } \frac{b}{\tau} \leq Z \leq \frac{b}{\tau} + \varepsilon \right\} \leq 2\varepsilon,
\]
we arrive at
\[
\lim_{\tau \to \infty} \mathbb{E} \left[ \frac{1}{1 + b \rho''(\text{prox}_{b\rho}(\tau Z))} \right] - \lim_{\tau \to \infty} \mathbb{P} \left\{ Z < 0 \text{ or } Z > \frac{b}{\tau} \right\} \leq 6\varepsilon.
\]

Since \( \varepsilon > 0 \) can be arbitrarily small, we have
\[
\lim_{\tau \to \infty} \mathbb{E} \left[ \frac{1}{1 + b \rho''(\text{prox}_{b\rho}(\tau Z))} \right] = \lim_{\tau \to \infty} \mathbb{P} \left\{ Z < 0 \text{ or } Z > \frac{b}{\tau} \right\}
\]
(15)
when \( b = c_0 \tau \). Similarly,
\[
\lim_{\tau \to \infty} \mathbb{E} \left[ (\rho'(\text{prox}_{b\rho}(\tau Z)))^2 \right] = \lim_{\tau \to \infty} \left\{ \mathbb{P} \left\{ Z > \frac{b}{\tau} \right\} + \frac{\tau^2}{b^2} \mathbb{E} \left[ Z^2 \mathbb{1}_{\{0 < Z < \frac{b}{\tau}\}} \right] \right\}
\]
\]
1.2 The probit case

The proof proceeds with the following 3 steps:

(i) Show that for any $b > 0$ and $\epsilon > 0$, there exist constants $c_{1,b}$, $c_{2,b}$, $c_3$, $c_4 > 0$, depending on $\epsilon$, such that

\[
\begin{align*}
&\left\{ \sup_{z > c_{1,b}} \left| \text{prox}_{b\varphi}(z) - \frac{z}{b+1} \right| \right\} \leq \epsilon, \\
&\left\{ \sup_{z < -c_{2,b}} \left| \text{prox}_{b\varphi}(z) - z \right| \right\} \leq \epsilon,
\end{align*}
\]

and

\[
\begin{align*}
&\left\{ \sup_{z > c_3} \left| \rho''(z) - 1 \right| \right\} \leq \epsilon, \\
&\left\{ \sup_{z < -c_4} \left| \rho''(z) \right| \right\} \leq \epsilon.
\end{align*}
\]

In particular, one can take

\[
c_{1,b} := \max \left\{ b\varphi'(\sqrt{2}) + \sqrt{2}, \ 2\sqrt{2b}, \ \frac{4b}{\epsilon} \right\}, \quad \text{and} \quad c_{2,b} := \max \left\{ 2b\varphi'(0), \ \sqrt{8 \log \frac{b}{\epsilon}} \right\}.
\]

(ii) Show that for any constant $\eta > 0$, for all $\tau$ sufficiently large, one has

\[
\left| 1 - \frac{1}{b(\tau) + 1} - 2\kappa \right| \leq \eta.
\]

(iii) Show that for any constant $0 < \eta < 1 - 2\kappa$ and for $\tau$ sufficiently large, one has

\[
\left| \frac{\sqrt{\tau^2}}{\tau^2} - 2\kappa \right| \leq \eta.
\]

In the sequel, we elaborate on each of these three steps.

Step (i). Recall that for any $x > 0$, one has $\frac{\phi(x)}{x} \left( 1 - \frac{1}{x} \right) \leq 1 - \Phi(x) \leq \frac{\phi(x)}{1 - \Phi(x)}$. Since $\rho'(x) = \frac{\phi(x)}{1 - \Phi(x)}$, this gives

\[
|\rho'(x) - x| \leq \frac{1}{x - x^{-1}} \leq \frac{2}{x}, \quad x \geq \sqrt{2}.
\]

We start with the first inequality in (16). From the definition of $\text{prox}(\cdot)$, we have the defining relation

\[
b\varphi'(\text{prox}_{b\varphi}(z)) + \text{prox}_{b\varphi}(z) = z.
\]

Therefore, if we take $z_{b,1} := b\varphi'(\sqrt{2}) + \sqrt{2}$, then this identity (21) indicates that $\text{prox}_{b\varphi}(z_{b,1}) = \sqrt{2}$. Moreover, $\text{prox}_{b\varphi}(z)$ is monotonically increasing in $z$ (see [2, Eqn. (56)]), which tells us that

\[
\text{prox}_{b\varphi}(z) \geq \text{prox}_{b\varphi}(z_{b,1}) = \sqrt{2}, \quad \forall z > z_{b,1}.
\]

Rearranging the identity (21) and combining it with (20) and (22), we obtain

\[
z - (b + 1)\text{prox}_{b\varphi}(z) = b\varphi'(\text{prox}_{b\varphi}(z)) - b\text{prox}_{b\varphi}(z)
\]

\[
\implies \left| \frac{z}{b+1} - \text{prox}_{b\varphi}(z) \right| = \left| b \frac{\rho'(\text{prox}_{b\varphi}(z)) - \text{prox}_{b\varphi}(z)}{b+1} \leq \frac{2b}{b+1}\text{prox}_{b\varphi}(z) \right| \leq \frac{2b/(b+1)}{\text{prox}_{b\varphi}(z)}
\]

\[
\leq \sqrt{2b}, \quad \forall z > z_{b,1}.
\]

This inequality provides a lower bound on $\text{prox}_{b\varphi}(z)$:

\[
\text{prox}_{b\varphi}(z) \geq \frac{z - \sqrt{2b}}{b+1} \geq \frac{z}{2(b+1)}
\]

for all $z$ obeying $z > z_{b,1}$ and $z > 2\sqrt{2b}$. Substitution into (23) once again gives

\[
\left| \frac{z}{b+1} - \text{prox}_{b\varphi}(z) \right| \leq \frac{2b/(b+1)}{\text{prox}_{b\varphi}(z)} \leq \frac{4b}{z} \leq \epsilon, \quad \forall z > \max \left\{ z_{b,1}, \ 2\sqrt{2b}, \ \frac{4b}{\epsilon} \right\},
\]
establishing the second bound of (16).

We now turn to the second result in (16). Similarly, it is seen from (21) that \( \prox_{b\rho}(z_{b,2}) = 0 \) with \( z_{b,2} := b\rho'(0) > 0 \). The monotonicity of \( \prox_{b\rho}(\cdot) \) implies that
\[
\prox_{b\rho}(z) \leq \prox_{b\rho}(z_{b,2}) = 0, \quad \forall z < z_{b,2}.
\]

Recognizing that \( \rho'(x) > 0 \) and \( \rho''(x) > 0 \) for any \( x \) and using the relation (21), we arrive at
\[
|z - \prox_{b\rho}(z)| = b\rho'(\prox_{b\rho}(z)) \leq b\rho'(0), \quad \forall z < z_{b,2},
\]
thus indicating that
\[
\prox_{b\rho}(z) \leq z + b\rho'(0) \leq z/2, \quad \forall z < -2z_{b,2} < 0.
\]

Substituting it into (25) and using the fact that \( \rho'(x) = \frac{\phi(x)}{1 - \Phi(x)} \leq 2\phi(x) \leq e^{-x^2/2} \) for all \( x < 0 \), we get
\[
|z - \prox_{b\rho}(z)| = b\rho'(\prox_{b\rho}(z)) \leq (a) b\rho'(z/2) \leq be^{-z^2/8}, \quad \forall z < -2z_{b,2} < 0,
\]
where (a) follows since \( \rho''(x) > 0 \). The upper bound (26) will not exceed \( \epsilon > 0 \) as long as \( z < -\max \left\{ 2z_{b,2}, \sqrt{8 \log \frac{1}{\delta}} \right\} \).

This establishes the second bound of (16).

The remaining two inequalities regarding \( \rho'' \) are rather straightforward and the proofs are thus omitted.

**Step (ii).** Recognizing that \( \Psi'(z;b) = \frac{b\rho'(z)}{1 + b\rho''(z)} \big|_{z=\prox_{b\rho}(z)} \), we see that \( b(\tau) \) is the solution to
\[
1 - \kappa = \E[g(\tau Z, b)] \quad \text{with} \quad g(x, b) := \frac{1}{1 + b\rho''(\prox_{b\rho}(x))}.
\]

As a result, everything boils down to quantifying \( \E[g(\tau Z, b)] \).

Consider any sufficiently small \( \epsilon > 0 \). We first obtain an approximation of \( \E[g(\tau Z, b)] \). Specifically, we claim that taking \( c_{\epsilon} := \frac{1}{2}\tau \epsilon^2 \) leads to
\[
\E \left[ g(\tau Z, b) 1_{|\tau Z| > c_{\epsilon}} \right] \leq \E[g(\tau Z, b)] \leq \E \left[ g(\tau Z, b) 1_{|\tau Z| > c_{\epsilon}} \right] + \epsilon.
\]

The lower bound is trivial since \( 0 \leq g(x, b) \leq 1 \). To see why the upper bound holds, we invoke Cauchy-Schwarz to derive
\[
\E \left[ g(\tau Z, b) 1_{|\tau Z| \leq c_{\epsilon}} \right] \leq \sqrt{\E \left[ g^2(\tau Z, b) \right]} \sqrt{\P \left( |\tau Z| \leq \frac{c_{\epsilon}}{\tau} \right) \leq \sqrt{\P \left( |\tau Z| \leq \frac{c_{\epsilon}}{\tau} \right) \leq \sqrt{2\frac{c_{\epsilon}}{\tau} = \epsilon}},
\]
where (b) arises since \( 0 \leq g(x, b) \leq 1 \). This inequality (29) matches the upper bound in (28). In short, we see that \( \E \left[ g(\tau Z, b) 1_{|\tau Z| > c_{\epsilon}} \right] \) is a reasonably tight approximation of \( \E \left[ g(\tau Z, b) \right] \), and it suffices to look at
\[
\E \left[ g(\tau Z, b) 1_{|\tau Z| > c_{\epsilon}} \right] = \E \left[ g(\tau Z, b) 1_{|\tau Z| < -c_{\epsilon}} \right] + \E \left[ g(\tau Z, b) 1_{|\tau Z| > c_{\epsilon}} \right].
\]

We first control the second term in the right-hand side of (30). Suppose for the moment that
\[
c_{\epsilon} > \max \left\{ c_{1,b}, (c_3 + \epsilon)(b + 1), c_{2,b} + c_4 + \epsilon \right\}.
\]

According to (10), on the event \( \{\tau Z > c_{\epsilon}\} \) one has
\[
\frac{\tau Z}{b + 1} - \epsilon \leq \prox_{b\rho}(\tau Z) \leq \frac{\tau Z}{b + 1} + \epsilon \quad \text{and} \quad 1 - \epsilon \leq \rho''(\prox_{b\rho}(\tau Z)) \leq 1 + \epsilon,
\]
where the second inequality holds since \( \prox_{b\rho}(\tau Z) \geq \frac{\tau Z}{b + 1} - \epsilon > \frac{c_{\epsilon}}{b + 1} - \epsilon \geq c_3 \). Plugging these inequalities into (27) gives
\[
\frac{1}{1 + b(1 + \epsilon)} \leq g(\tau Z, b) \leq \frac{1}{1 + b(1 - \epsilon)}.
\]

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In addition, similar to (29) we get
\[ \frac{1}{2} \geq \mathbb{P}(\tau Z > c_\epsilon) = \mathbb{P}(\tau Z < -c_\epsilon) = \frac{1}{2} \left\{ 1 - \mathbb{P}\left( |Z| \leq \frac{c_\epsilon}{\tau} \right) \right\} \geq \frac{1}{2} \left\{ 1 - \frac{2c_\epsilon}{\tau} \right\} = \frac{1}{2} (1 - \epsilon^2). \]

The above bounds taken collectively reveal that
\[ \frac{1}{1 + b(1 + \epsilon)} : \frac{1}{2} (1 - \epsilon^2) \leq \mathbb{E} \left[ g(\tau Z, b) 1_{|\tau Z| > c_\epsilon} \right] \leq \frac{1}{1 + b(1 - \epsilon)} : \frac{1}{2}. \]  

(31)

We can employ similar arguments to control the first term in the right-hand side of (28) as well. Since \( c_\epsilon > \max\{c_2, c_4 + \epsilon\} \), on the event \( \{\tau Z < -c_\epsilon\} \) we have
\[ \tau Z - \epsilon \leq \text{prox}_b(\tau Z) \leq \tau Z + \epsilon \quad \text{and} \quad \epsilon \leq \rho''(\text{prox}_b(\tau Z)) \leq \epsilon, \]
a direct consequence of (16). This implies that
\[ \frac{1}{1 + b(1 - \epsilon)} \leq g(\tau Z, b) \leq \frac{1}{1 - b\epsilon} \]
and, therefore,
\[ \frac{1}{1 + b(1 - \epsilon)} : \frac{1}{2} (1 - \epsilon^2) \leq \mathbb{E} \left[ g(\tau Z, b) 1_{|\tau Z| < c_\epsilon} \right] \leq \frac{1}{1 - b\epsilon} : \frac{1}{2}. \]  

(32)

Combining (28), (31) and (32), we conclude that for any \( \epsilon > 0 \),
\[ \frac{1 - \epsilon^2}{2} \left\{ \frac{1}{1 + b(1 + \epsilon)} + \frac{1}{1 + b} \right\} \leq \mathbb{E} \left[ g(\tau Z, b) \right] \leq \frac{1}{2} \left\{ \frac{1}{1 + b(1 - \epsilon)} + \frac{1}{1 - b\epsilon} \right\} + \epsilon, \]
as long as \( c_\epsilon = \frac{1}{2} \tau \epsilon^2 > \max\{c_{1,b}, (c_3 + \epsilon)(b + 1), c_{2,b}, c_4 + \epsilon\} \), or equivalently,
\[ \tau > \frac{2 \max\{c_{1,b}, (c_3 + \epsilon)(b + 1), c_{2,b}, c_4 + \epsilon\}}{\epsilon^2}, \]
where the lower bound is on the order of \( b/\epsilon^3 \). Effectively, we have established that for any given \( b \) and any sufficiently small \( \epsilon > 0 \) (so that \( b\epsilon < 1 \) and \( \epsilon < 1 \)), if \( \tau \) is sufficiently large (as specified above) one has
\[ \left| \mathbb{E} \left[ g(\tau Z, b) \right] - \frac{1}{2} \left( \frac{1}{1 + b} + 1 \right) \right| \leq \tilde{c}_4 (\epsilon + b\epsilon) \]  

(33)

for some universal constant \( \tilde{c}_4 > 0 \) independent of \( b, \epsilon, \tau \).

We can then combine this result (33) with the constraint (27) to derive an estimate on \( b(\tau) \). Fix any \( \eta > 0 \). Let \( b_1 \) and \( b_2 \) be two constants such that
\[ \frac{1}{2} \left( \frac{1}{1 + b_1} + 1 \right) = 1 - \kappa - \frac{\eta}{4}, \quad \frac{1}{2} \left( \frac{1}{1 + b_2} + 1 \right) = 1 - \kappa + \frac{\eta}{4}. \]

Picking \( \epsilon > 0 \) sufficiently small so that \( \max\{\frac{\epsilon^2}{2}, \frac{\epsilon}{2}\} \leq \frac{\eta}{4} \) and \( \tau \gg \max\{b_1, b_2\}/\epsilon^3 \), we can ensure that
\[ \mathbb{E} \left[ g(\tau Z, b_1) \right] < 1 - \kappa < \mathbb{E} \left[ g(\tau Z, b_2) \right]. \]

Recall that for any \( \tau > 0 \), the function \( G(b) := 1 - \mathbb{E} [g(\tau Z, b)] \) is strictly increasing in \( b \) (see [1] Lemma 5) and, hence,
\[ b_2 \leq b(\tau) \leq b_1, \quad \implies \quad \frac{1}{2(1 + b_1)} \leq \frac{1}{2(1 + b(\tau))} \leq \frac{1}{2(1 + b_2)}. \]

Combining these together, we obtain
\[ \left| \left( 1 - \frac{1}{b(\tau) + 1} \right) - 2\kappa \right| \leq \eta, \]  

(34)
for any \( \eta > 0 \) with the proviso that \( \tau \) is sufficiently large. This finishes Step (ii). In particular, this yields
\[
\lim_{\tau \to \infty} b(\tau) = \frac{2\kappa}{1 - 2\kappa}.
\] (35)

**Step (iii).** Now we move on to the variance map
\[
\mathcal{V}(\tau^2) = \frac{b(\tau)^2}{\kappa} \mathbb{E} \left[ \rho'(\text{prox}_{b\rho}(\tau Z))^2 \right].
\] (36)

For notational convenience, we set
\[
h(x) := \rho'(\text{prox}_{b\rho}(x))^2,
\] a key mapping in the definition (36). Before proceeding, we remark that from the properties of \( \rho' \), for any \( \epsilon > 0 \), there exist constants \( c_5, c_6 > 0 \), depending on \( \epsilon \), such that
\[
\sup_{z > c_5} |\rho'(z) - z| \leq \epsilon, \quad \sup_{z < -c_6} |\rho'(z)| \leq \epsilon.
\] (37)

As before, we decompose the function \( \mathcal{V}(\tau^2) \) as follows:
\[
\left| \mathcal{V}(\tau^2) - \frac{b(\tau)^2}{\kappa} \mathbb{E} \left[ h(\tau Z) \mathbf{1}_{\{\tau Z < \alpha_\epsilon\}} \right] \right| = \frac{b(\tau)^2}{\kappa} \mathbb{E} \left[ h(\tau Z) \mathbf{1}_{\{\tau Z \leq \alpha_\epsilon\}} \right]
\] for some point \( \alpha_\epsilon > 0 \) to be specified later. This gives
\[
\mathbb{E}[h(\tau Z) \mathbf{1}_{\{\tau Z < \alpha_\epsilon\}}] \leq \sqrt{\mathbb{E}[h^2(\tau Z) \mathbf{1}_{\{\tau Z < \alpha_\epsilon\}}]} \sqrt{\mathbb{P}(\{\tau Z \leq \alpha_\epsilon\})} \leq C(\alpha_\epsilon, b) \sqrt{2\Phi \left( \frac{\alpha_\epsilon}{\tau} \right)} - 1,
\] (38)
where
\[
C(\alpha_\epsilon, b) = \rho'((\text{prox}_{b\rho}(\alpha_\epsilon))^2).
\]
The last inequality of (38) holds since (1) \( \rho'(z) \geq 0 \) is an increasing function of \( z \); (2) \( \text{prox}_{b\rho}(x) \) is an increasing function of \( x \) (see [2, Eqn. (56)]). For any given \( \epsilon > 0 \), one can pick \( \tau \) sufficiently large so that the above bound \( C(\alpha_\epsilon, b) \sqrt{2\Phi \left( \frac{\alpha_\epsilon}{\tau} \right)} - 1 \) is below \( \epsilon \). The particular choice of \( \tau \) will be made clear later. Under these conditions,
\[
\mathbb{E}[h(\tau Z) \mathbf{1}_{\{\tau Z > \alpha_\epsilon\}}] \leq \mathbb{E}[h(\tau Z)] \leq \mathbb{E}[h(\tau Z) \mathbf{1}_{\{\tau Z < -\alpha_\epsilon\}}] + \mathbb{E}[h(\tau Z) \mathbf{1}_{\{\tau Z > \alpha_\epsilon\}}] + \epsilon.
\] (39)

We first control the second term in the right-hand side of (39). To this end, we choose
\[
\alpha_\epsilon > \max \{c_1, b, c_2, b, (c_5 + \epsilon)(b + 1), c_6 + 2\epsilon\}
\] as before. Then from (16) and (37), on the event \( \{\tau Z > \alpha_\epsilon\} \) we have
\[
\frac{\tau Z}{b + 1} - \epsilon \leq \text{prox}_{b\rho}(\tau Z) \leq \frac{\tau Z}{b + 1} + \epsilon \quad \text{and} \quad \frac{\tau Z}{b + 1} - 2\epsilon \leq \rho'(\text{prox}_{b\rho}(\tau Z)) \leq \frac{\tau Z}{b + 1} + 2\epsilon.
\] This yields
\[
\left( \frac{\tau Z}{b + 1} - 2\epsilon \right)^2 \leq h(\tau Z) \leq \left( \frac{\tau Z}{b + 1} + 2\epsilon \right)^2
\] on the event \( \{\tau Z > \alpha_\epsilon\} \), and hence
\[
\mathbb{E} \left[ \left( \frac{\tau Z}{b + 1} - 2\epsilon \right)^2 \mathbf{1}_{\{\tau Z > \alpha_\epsilon\}} \right] \leq \mathbb{E}[h(\tau Z) \mathbf{1}_{\{\tau Z > \alpha_\epsilon\}}] \leq \mathbb{E} \left[ \left( \frac{\tau Z}{b + 1} + 2\epsilon \right)^2 \mathbf{1}_{\{\tau Z > \alpha_\epsilon\}} \right].
\] (40)

Similarly for the first term in the right-hand side of (39), as \( \alpha_\epsilon > \max \{c_2, c_6 + 2\epsilon\} \), on the event \( \{\tau Z < -\alpha_\epsilon\} \), we have
\[
\tau Z - \epsilon \leq \text{prox}_{b\rho}(\tau Z) \leq \tau Z + \epsilon \quad \text{and} \quad -\epsilon \leq \rho'(\text{prox}_{b\rho}(\tau Z)) \leq \epsilon.
\]
Note that $P(\tau Z > \alpha) = P(\tau Z < -\alpha) = \frac{1}{2}(1 - \delta_\epsilon)$ for some $\delta_\epsilon$ small which is a function of $\epsilon$ and which vanishes as $\epsilon \to 0$. This yields

$$0 \leq E[h(\tau Z)1_{\{\tau Z < -\alpha\}}] \leq \frac{\epsilon^2}{2}(1 - \delta_\epsilon).$$

Combining the relations (39), (40) and (41) we obtain that

$$\frac{b^2}{\kappa} E \left[ \left( \frac{\tau Z}{b + 1} - 2\epsilon \right)^2 1_{\{\tau Z > \alpha\}} \right] \leq \mathcal{V}(\tau^2) \leq \frac{b^2}{\kappa} \left\{ E \left[ \left( \frac{\tau Z}{b + 1} + 2\epsilon \right)^2 1_{\{\tau Z > \alpha\}} \right] + \frac{\epsilon^2}{2}(1 - \delta_\epsilon) + \epsilon \right\}. \quad (42)$$

We still need to evaluate $E \left[ \left( \frac{\tau Z}{b + 1} - 2\epsilon \right)^2 1_{\{\tau Z > \alpha\}} \right]$. To this end, we define two quantities

$$\alpha_1 := E \left[ Z 1_{\{\tau Z > \alpha\}} \right] \quad \text{and} \quad \alpha_2 := E \left[ Z^2 1_{\{\tau Z > \alpha\}} \right].$$

Using the properties of the normal CDF, one can show that

$$\frac{\tau}{\sqrt{2\pi}} - \frac{\alpha_\epsilon}{2} \leq \tau \alpha_1 \leq \frac{\tau}{\sqrt{2\pi}} \quad \text{and} \quad \frac{\tau^2}{2} - \frac{\alpha_\epsilon^2}{2} \leq \tau^2 \alpha_2 \leq \frac{\tau^2}{2}. \quad (43)$$

Using the above relations and rearranging, the bounds in (42) can be rewritten as

$$\mathcal{V}(\tau^2) \geq \frac{b^2}{\kappa} \left[ \frac{\tau^2}{2(b + 1)^2} - \frac{\alpha_\epsilon^2}{2} \frac{\delta_\epsilon}{2} - \frac{4\epsilon}{\sqrt{2\pi}(b + 1)} + 2\epsilon^2(1 - \delta_\epsilon) \right];$$

$$\mathcal{V}(\tau^2) \leq \frac{b^2}{\kappa} \left[ \frac{\tau^2}{2(b + 1)^2} + \epsilon \left( \frac{4\pi}{\sqrt{2\pi}(b + 1)} + 1 \right) + \frac{5}{2} \epsilon^2(1 - \delta_\epsilon) \right].$$

Finally, observing that $b \geq 0$, we arrive at

$$\left| \mathcal{V}(\tau^2) - \frac{b^2}{2\kappa} \frac{\tau^2}{(b + 1)^2} \right| \leq \frac{b^2}{\kappa} \left\{ \epsilon \left( \frac{8\pi}{\sqrt{2\pi}} + 1 \right) + \frac{\delta_\epsilon \alpha_\epsilon^2}{2} + \frac{\epsilon^2}{2}(1 - \delta_\epsilon) \right\},$$

which is equivalent to

$$\left| \frac{\mathcal{V}(\tau^2)}{\tau^2} - \frac{1}{2\kappa} \left( 1 - \frac{1}{b + 1} \right) \right|^2 \leq \frac{b^2}{\kappa} \left\{ \epsilon \left( \frac{8\pi}{\sqrt{2\pi}} + 1 \right) + \frac{\delta_\epsilon \alpha_\epsilon^2}{2\tau^2} + \frac{\epsilon^2}{2\tau^2}(1 - \delta_\epsilon) \right\}. \quad (44)$$

Note that in the bound above $\alpha_\epsilon$ also depends on $b$. Henceforth we denote $\alpha_\epsilon$ as $\alpha_\epsilon(b)$. Next, we invoke the result from Step (ii) to ensure that $b(\tau)$ is bounded for all sufficiently large values of $\tau$.

Fix $\eta' > 0$ such that $0 < \eta' < 1 - 2\kappa$. Let $\tau_0$ be the threshold above which for all values of $\tau$ the relation (34) holds with $\eta = \eta'/2$. Then $\forall \tau \geq \tau_0$, one has $b(\tau) \leq \frac{2\kappa + \eta'}{1 - 2\kappa - \eta'} =: a(\eta')$.

For all $\tau \geq \tau_0$, we have

$$\left| \frac{\mathcal{V}(\tau^2)}{\tau^2} - \frac{1}{2\kappa} \left( 1 - \frac{1}{b + 1} \right) \right|^2 \leq \frac{a(\eta')^2}{\kappa} \left\{ \epsilon \left( \frac{8\pi}{\sqrt{2\pi}} + 1 \right) + \frac{\delta_\epsilon \alpha_\epsilon(a(\eta)))^2}{2\tau^2} + \frac{\epsilon^2}{2\tau^2}(1 - \delta_\epsilon) \right\},$$

where $\alpha_\epsilon(a(\eta))$ is any constant above $\max\{1, a(\eta)\epsilon, c_5 + \epsilon\}$. We choose $\tau > \tau_0$ so that $C(\alpha_\epsilon(a(\eta)), a(\eta))\sqrt{2\Phi(\alpha_\epsilon)} - \eta'$ is below $\epsilon$, and the above bound in the RHS is below $\eta = \eta'/2$. This gives

$$\left| \frac{\mathcal{V}(\tau^2)}{\tau^2} - \frac{1}{2\kappa} \left( 1 - \frac{1}{b + 1} \right) \right|^2 + \left| 2\kappa - \frac{1}{2\kappa} \left( 1 - \frac{1}{b + 1} \right) \right|^2 \leq \eta'.$$
Hence, for any such $\tau$

\[
\frac{\mathcal{V}(\tau^2)}{\tau^2} \leq 2\kappa + \eta' < 1,
\]

from the choice of $\eta'$. In particular, we have established that

\[
\lim_{\tau \to \infty} \frac{\mathcal{V}(\tau^2)}{\tau^2} = 2\kappa.
\]

**Remark 1.** In fact, the above analysis works for a broader class of link functions beyond the probit case. Specifically, more general sufficient conditions for the above result to hold are the following: in addition to conditions mentioned in [1, Section 2.3.3].

- $\rho'(x) \to 0$ when $x \to -\infty$, and $\rho'(x)/x \to 1$, when $x \to \infty$; further, $|\rho'(x) - x| \leq f(x)$ for all $x$ positive, where $f(x)$ is some function obeying $f(x) \to 0$ when $x \to \infty$.

- $\rho''$ is bounded, converges to 1 when $x \to \infty$ and converges to 0 when $x \to -\infty$. $-\infty$ are swapped.

- In addition, for any given $z$, $b\rho''(\text{prox}_{b\rho}(z)) \to \infty$ when $b \to \infty$.

**References**
