

# Curvelets and Reconstruction of Images from Noisy Radon Data

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## ABSTRACT

The problem of recovering an input signal from noisy and linearly distorted data arises in many different areas of scientific investigation; e.g., noisy Radon inversion (tomography) is a problem of special interest and considerable practical relevance in medical imaging. We will argue that traditional methods for solving inverse problems – damping of the singular value decomposition (SVD) or cognate methods – behave poorly when the object to recover has edges.

We apply a new system of representation, namely, *the curvelets* in this setting. Curvelets provide near-optimal representations of otherwise smooth objects with discontinuities along smooth  $C^2$  edges. Inspired by some recent work on nonlinear estimation, we construct a curvelet-based biorthogonal decomposition of the Radon operator and build a reconstruction based on the shrinkage (or thresholding) of the noisy curvelet coefficients. This novel approach is shown to give a new theoretical understanding of the problem of edges in the Radon inversion problem.

**Keywords:** Ill-Posed Inverse Problems, Regularization, Singular Value Decomposition, Radon Transform, Deconvolution, Edge, Edge-Preserving Regularization, Wavelet-Vaguelette-Decomposition.

## 1. INTRODUCTION

A classical problem is the reconstruction of images from indirect noisy measurements; that is, we wish to recover an object  $f \in L_2(\mathbb{R}^2)$  from data  $y$  of the form

$$y(u) = (Kf)(u) + z(u), \quad u \in \mathcal{U}; \quad (1)$$

$f(x)$  is the object of interest;  $K$  is a linear operator (we are able to observe data about  $(Kf)(u)$  only)

$$(Kf)(u) = \int k(u, x)f(x) dx,$$

$K$  might be a convolution transform, a Radon transform or an Abel transform;  $z$  is a noise contribution that might be stochastic or deterministic. Such problems are called *linear inverse problems* and their study is a very active area of scientific investigation. Singular value decomposition (SVD) methods constitute *the* standard approach for linear inverse problems. However, there is a conceptual problem associated with this approach: the SVD is driven by the structure of the forward operator, with no input from the structure of the object to be recovered. A byproduct is that the standard approach is suboptimal for recovering spatially inhomogeneous objects.

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Although our theory supports general homogeneous or quasi-homogeneous operators, we will consider the case where the linear operator is the Radon transform, whose inversion has for instance wide ranging applications in tomography. The Radon transform of  $f$  is the collection of line integrals

$$(Kf)(u) = (Rf)(\theta, t) = \int_{\mathcal{L}_{\theta,t}} f(x) dx,$$

where  $\mathcal{L}_{\theta,t}$  is the line defined by  $\{x_1 \cos \theta + x_2 \sin \theta = t\}$ . The Radon inversion problem arises in various fields of scientific inquiry like medical or seismic imaging.

### 1.1. Edge-Preserving Regularization

Images exhibit edges and recovering those edges is a very important topic in many scientific applications. In biological imagery, say, edges may indicate the boundary of a tumor and, therefore, it is essential to gather information about their locations and sizes as precisely as possible. The goal is then to develop reconstruction techniques that would smooth out the ‘flat’ part of the image under study without blurring the edges. The importance of edges is illustrated by the fact that in tomography for instance, reconstruction of images are tested on piecewise constant images –the popular phantoms. Basically, for such test images, the only information to recover is contained in the edges.

Motivated by problems of considerable practical relevance, the existing literature of edge-preserving deconvolution/ tomography is immense. A partial listing of articles specifically devoted to this theme would include Refs. 1–8. This is an imaginative literature without much fundamental knowledge, but with many experiments and much energy and enthusiasm. For example, while PDE methods and especially total-variation based image processing methods are claimed to be better than wavelets, there is no known theory that actually explains why this should be so.

### 1.2. Curvelets and Singularities Along Curves

In recent work,<sup>9,10</sup> we introduced tight frames of curvelets for representing arbitrary functions in  $L_2(\mathbb{R}^2)$ . Curvelets provide a new multiresolution representation with several features that set them apart from existing representations such as wavelets. The keypoint here is that there exists a profound connection between curvelets and edges. In fact curvelets were originally tailored for providing efficient representations of images with edges.

Let us consider an object  $f$  supported in  $[0, 1]^2$  which is smooth away from a discontinuity across a  $C^2$  curve. We may think of  $f$  as a sophisticated phantom; like phantoms,  $f$  is edge-dominated but it is not necessarily piecewise constant. The point is that curvelets provide an optimal representation of objects like  $f$ . Consider the class  $\mathcal{F}$  of all such objects –this class is of special interest to researchers working in the area of edge-preserving deconvolution/tomography. Then it is proved in Ref. 9 that the curvelet representation of the class  $\mathcal{F}$  is as sparse as possible. In other words, no basis gives a sparser or more ‘economical’ representation of our class  $\mathcal{F}$ .

In statistics and signal processing, the potential for sparsity has a well-known meaning for signal estimation; e.g., here, image restoration and reconstruction, see Ref. 11 for example. An optimally sparse representation concentrates the energy in as few coefficients as possible. This viewpoint is interesting because it effectively reduces the dimension of the object to be recovered: reconstructions based on the estimation of a relatively small number of coefficients (the largest ones) can be quite accurate. In many interesting situations however including the special situation of curvelet representations of objects with edges, the location of the big coefficients is not known a priori. This phenomenon calls for nonlinear procedures. This is in stark contrast with standard SVD methods, see section 2.1.

### 1.3. Curvelets and Radon Inversion

This paper applies curvelets to the problem of noisy Radon inversion, exploiting the sparsity of such representations. To deploy them in the Radon setting, we construct a curvelet-based biorthogonal decomposition of the Radon operator and build “curvelet shrinkage” estimators based on thresholding of the noisy curvelet coefficients.

We have two main objectives in this paper:

1. To explain that our curvelet-based method corresponds to a new way of processing Radon data, unlike any in practice.
2. To develop the idea that curvelets give a rigid analysis of objects which can be used to give a new theoretical understanding of the problem of edges in the Radon inversion problem.
  - (a) An analysis which is very amenable to rigorous proof. In the past, singular value decompositions gave rigorous results, and were suited to proofs, but they did not give much information about reconstructing edges.
  - (b) There is a beautiful representation of the inverse problem which shows that the analysis detects edges at certain locations and orientations in the Radon domain and automatically synthesizes edges at corresponding locations and directions in the original domain.

## 2. CURVELETS

Because of space limitation, we shall limit ourselves to listing a few properties of the curvelet frame. A detailed account of the curvelet construction is given in Ref. 9. Ref. 10 contains a rather coarser description. In addition, the authors signed another paper in these proceedings, namely, ‘Curvelets, Multiresolution Representation, and Scaling Laws,’ which also presents an overview of the curvelet system.

- There is a collection  $(\gamma_\mu)$  with  $\mu$  running through a discrete index set  $\mathcal{M}$  which makes a tight frame for  $L^2(\mathbb{R}^2)$ . This means there is a reproducing formula

$$f = \sum_{\mu} \langle f, \gamma_\mu \rangle \gamma_\mu$$

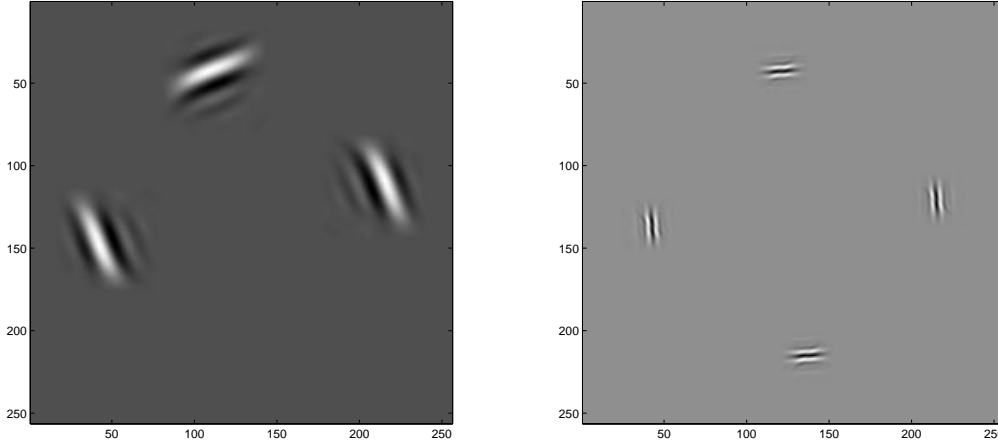
and a Parseval-type relation

$$\|f\|_{L^2(\mathbb{R}^2)}^2 = \sum_{\mu} |\langle f, \gamma_\mu \rangle|^2.$$

- The set  $\mathcal{M}$  has a seven-index structure  $\mu = (s, k_1, k_2; j, k; i, \ell, \epsilon)$  whose indices include parameters for *scale*, *location*, *direction*, and *microlocation*.
- The elements of the tight frame with substantial  $L^2$ -norm obey a special “anisotropic scaling law”: the width of the effective support of these elements is effectively proportional to the length *squared*. The frame elements become successively more anisotropic at progressively finer scales.
- The number of distinct directions at a given scale grows as  $scale^{-1}$ .

The last two properties are quite different from those of pre-existing multiscale representations such as wavelets, where the aspect ratios of basis/frame elements remain fixed as one moves to finer scales and the number of distinct directions remains fixed also.

Figure 1 displays a few curvelets at different scales, i.e. different values of the parameter  $s$ . We hope that they will provide the reader with an idea of the main geometric features of this new system.



**Figure 1.** Representation of a few curvelets at two different scales. The scale of the curvelets displayed on the left panel is coarser than the scale of those displayed on the right panel.

### 3. CURVELETS AND BCD

#### 3.1. SVD

The singular system decomposition is the well-established approach for solving problems of the form (1), see Refs. 12,13 for example. We let  $(e_\nu(x))$  denote the eigenfunctions of the Gram operator  $K^*K$  (we assume that  $K^*K$  is compact) and  $k_\nu^2$  its eigenvalues, i.e.  $K^*K e_\nu = k_\nu^2 e_\nu$ . Further, letting  $h_\nu$  be the normalized image of the eigenfunction  $e_\nu$  through the forward operator  $K$ ,  $K e_\nu = k_\nu h_\nu$ , with  $\|h_\nu\| = 1$ , we have available the following reproducing formula

$$f = \sum k_\nu^{-1} [Kf, h_\nu] e_\nu, \quad (2)$$

whenever none of the eigenvalues is zero;  $[, ]$  stands for the inner product of  $L_2(du)$  while in the remainder of the paper, the notation  $\langle, \rangle$  will refer to the inner product of  $L_2(dx)$ . The SVD explains the possibly ill-posedness of the problem. In many interesting situations (including the case where  $K$  is the Radon transform)  $k_\nu \rightarrow 0$  when  $\nu \rightarrow \infty$ .

It would be unrealistic to substitute  $Kf$  in the formula (2) with the observed data  $y$  as the noise would get amplified by a factor tending to infinity. A possible solution is to downweight the coefficients in the expansion associated with small values of  $k_\nu$ ; picking weights that are large for large values of  $k_\nu$  and small for small values  $k_\nu$ , we get a windowed SVD reconstruction

$$\hat{f}_w = \sum w_\nu k_\nu^{-1} [y, h_\nu] e_\nu;$$

for instance, one should pick the weights such that the reconstruction  $\hat{f}_w$  has at least finite variance. The choice of window has been the subject of some extensive development and special cases of windowed SVD include the method of LandWeber Strand, the method of quadratic regularization and the iterative damped backprojection.

#### 3.2. Curvelets and BCD

Roughly speaking, the singular system for the Radon inversion problem is composed of sinusoids  $e_\nu(x)$  with the double  $\nu = (\nu_1, \nu_2)$ ,  $\nu_1, \nu_2 \in \mathbb{Z}$ , indexing the two-dimensional frequencies and their eigenvalues obey  $|k_\nu| \sim |\nu|^{-1/2}$ . The windowed SVD paradigm then suggests damping high-frequencies. Unfortunately,

images with edges are only slowly decaying in frequency-space. In less sophisticated language, one may say that, because of edges, a significant fraction of the energy of the image is located at very high-frequency. From a reconstruction viewpoint, this says that desperately many terms are necessary for reconstructing an edge with good accuracy. Both from a theoretic and practical viewpoint, linear procedures such as the windowed SVD approach are do not perform very well as they tend to smooth out the edges. In this direction, very concrete mathematical results are given in Ref. 14.

Donoho<sup>15</sup> then observed that the singular system approach is entirely driven by the operator  $K$  and has very little to do with the object  $f$  one attempts to recover. He suggested replacing the singular system by any representation system that would simultaneously achieve two goals:

1. the new system would nearly diagonalize the Gram operator  $K^*K$ , and
2. the new system would provide a sparse representation of the class of objects one wish to reconstruct.

He further developed the Wavelet-Vaguelette-Decomposition where this paradigm is applied to the inversion of homogeneous or quasi-homogeneous operators  $K$  using wavelet bases.

We present the Operator-Biorthogonal Curvelet Decomposition (BCD) for the Radon operator. Similar decompositions are available, however, for arbitrary homogeneous or quasi-homogeneous operators  $K$ . The BCD is based on the curvelet frame and obeys the same philosophy as the WVD. It relies on

1. the fact that the curvelet frame nearly diagonalizes the Gram operator  $K^*K$ , and
2. the fact that the curvelet frame provides a sparse representation of otherwise smooth functions with discontinuities along curves, i.e. images with edges.

We first recall that curvelets provide a decomposition of any object in  $L_2(\mathbb{R}^2)$  of the form

$$f = \sum_{\mu} \langle f, \gamma_{\mu} \rangle \gamma_{\mu}.$$

We next introduce quasi-singular value relations: that is, putting  $\kappa_s = 2^{-s}$ , we construct companions  $(u_{\mu}), (v_{\mu})$  to the curvelet frame implicitly defined –in the Radon domain– by the following relations:

$$[Rf, u_{\mu}] = \kappa_s \langle f, \gamma_{\mu} \rangle \tag{3}$$

and

$$R\gamma_{\mu} = \kappa_s v_{\mu}, \quad R^*u_{\mu} = \kappa_s \gamma_{\mu}; \tag{4}$$

In the SVD approach the coefficient  $\langle f, e_{\nu} \rangle$  could be calculated via the identity  $\langle f, e_{\nu} \rangle = k_{\nu}^{-1} [Kf, h_{\nu}]$ . Likewise, the curvelet coefficient  $\langle f, \gamma_{\mu} \rangle$  of an object  $f$  –up to a multiplicative constant– is obtained by simply integrating the Radon transform  $Rf$  against the companion  $u_{\mu}$ ; the  $\kappa_s$ ' play the role of quasi-singular values so that the  $u_{\mu}$ 's are approximatively of unit norm. With these notations, the BCD is the decomposition of the Radon transform given by

$$f = \sum_{\mu} [Rf, u_{\mu}] \kappa_s^{-1} \gamma_{\mu}. \tag{5}$$

### 3.3. Details

We recall a few facts about the Radon transform.<sup>16</sup> First, letting  $\Delta$  be the Laplacian operator, it is well-known that  $R^*R = (-\Delta)^{-1/2}$ ; i.e., viewed in Fourier space,  $R^*R$  is a multiplication by the inverse of the modulus of the frequency. Second, put

$$\tilde{R} = (D^{1/2} \otimes I)R$$

where  $\{(D^{1/2} \otimes I)g\}(t, \theta)$  is the fractional derivative of order 1/2 acting on the  $t$ -variable (and the identity  $I$  on the angular variable  $\theta$ ). Then  $\tilde{R}$  is an isometry and

$$[\tilde{R}f, \tilde{R}g] = \langle f, g \rangle.$$

Define now  $(u_\mu)$  and  $(v_\mu)$  via

$$u_\mu = \tilde{R}c_\mu^+, \quad v_\mu = \tilde{R}c_\mu^-.$$

It is easy to verify that both sequences  $(u_\mu)$  and  $(v_\mu)$  obey the relations (3) and (4) introduced in the previous section. The BCD is introduced in far greater details in Ref. 14.

### 3.4. The Dual Curvelets

The pair  $(u_\mu, v_\mu)$  provides a system of analysis and synthesis of ‘sinograms.’ For instance, pushing the decomposition (5) through the Radon operator gives

$$Rf = \sum_{\mu} [Rf, u_\mu] v_\mu,$$

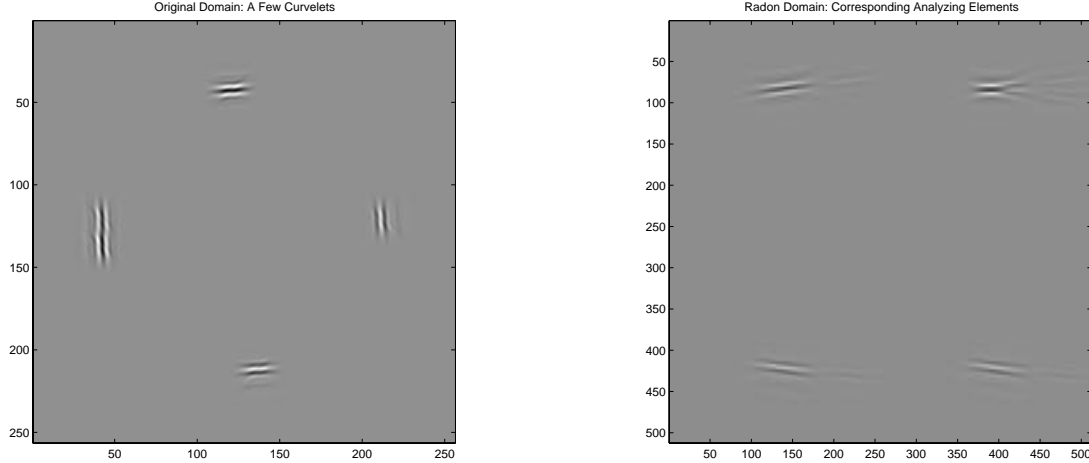
with equality holding in an  $L_2$  sense. In addition, the decomposition is stable in the sense that for any  $g \in L_2(dtd\theta)$ , we have

$$\|g\|_{L_2(dtd\theta)}^2 \sim \sum_{\mu} [g, u_\mu]^2,$$

and similarly for  $v_\mu$  (the symbol  $\sim$  means that the ratio is bounded away from zero and infinity). The pair  $(u_\mu), (v_\mu)$  is then said to build a system of dual frames for  $L_2(dtd\theta)$ .<sup>17</sup> Furthermore, they obey quasi-biorthogonality relations

$$[u_\mu, v_{\mu'}] = 2^{s'-s} \langle \gamma_\mu, \gamma_{\mu'} \rangle.$$

In the  $(t, \theta)$ -plane, both the elements  $u_\mu$  and  $v_\mu$  have the visual appearance of curvelets. They have a scale, an orientation, obey a scaling law, etc. Hence, they are especially adapted for the analysis and synthesis of edges in the Radon domain. In fact, the next section will develop a micro-local there is a microlocal correspondence between the dual system and the curvelet frame. A few of these dual elements are displayed on Figure 2.



**Figure 2.** The left panel represents a few curvelets  $\gamma_\mu$  at a given scale. The right panel displays the associated dual curvelets  $v_\mu$  in the Radon domain. The dual system shares the same features as the original system: i.e., scale, direction, location, anisotropy, etc.

#### 4. MICRO-LOCAL CORRESPONDENCE

In this section, we develop the concept of ‘micro-local’ correspondence. Microlocal correspondence is the idea that the dual curvelets pick up information from certain places/ directional orientations in the Radon domain and use that to reconstruct at certain places/directional orientations in the original domain.

The decomposition formula (5) directly associates behavior in the Radon domain with behavior in the object domain. The Radon domain data are analyzed by a bank of functions ( $u_\mu$ ) and each output coefficient scales a corresponding synthesized behavior  $\gamma_\mu$  in the object domain. Since the  $\gamma_\mu$  correspond at fine scales to highly localized directionally-oriented elements, the formula may be said at fine scales to be reading off the existence of edges at certain locations and orientations in the object domain from behavior of the Radon transform.

The correspondence between curvelets and dual curvelets as given in the reproducing formula has, at fine scales, an explicit geometric description. Let us consider a curvelet localized near spatial position  $x_0$  and codirection  $\theta_0$ ; the curvelet is aligned with the line

$$(x_1 - x_{0,1}) \cos \theta_0 + (x_2 - x_{0,2}) \sin \theta_0 = 0.$$

Then the corresponding dual curvelet is localized in the Radon plane at  $(t_0, \theta_0)$  and with direction  $\tau_0$ , where

$$t_0(x_0, \theta_0) = x_{0,1} \cos(\theta_0) + x_{0,2} \sin(\theta_0) \tag{6}$$

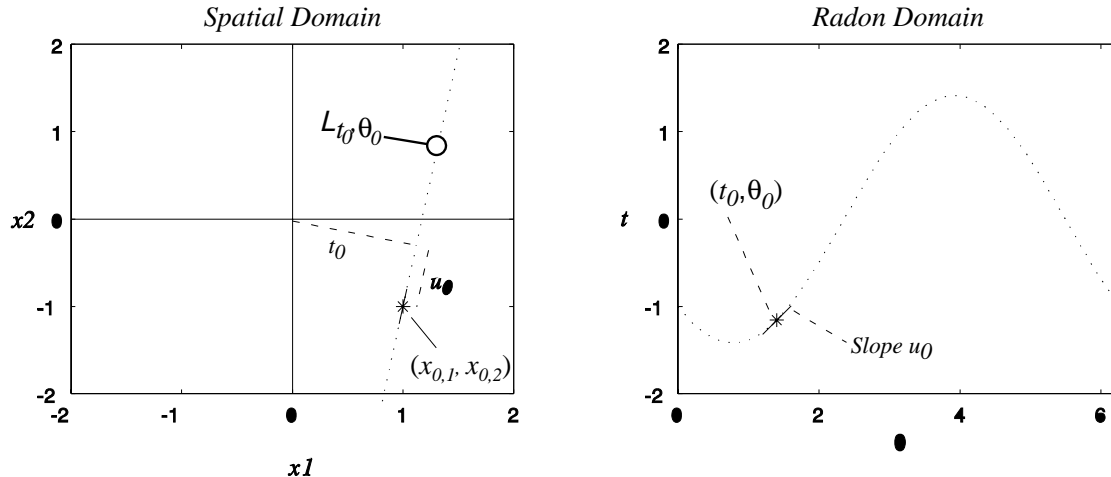
and

$$\tau_0(x_0, \theta_0) = \tan^{-1} [-x_{0,1} \sin(\theta_0) + x_{0,2} \cos(\theta_0)]. \tag{7}$$

This correspondence is illustrated on the diagram 3.

Pushing further the analysis, one comes to realize that the Radon transform maps edges in the original domain into edges in the Radon domain. To understand this point, let  $f$  be the indicator function of the unit disk

$$f(x) = 1_{\{|x| \leq 1\}}.$$



**Figure 3.** Correspondence between spatial domain and Radon domain. A curvelet localized near  $x_0 = (x_{0,1}, x_{0,2})$  in the spatial domain and oriented in direction  $\theta_0$ , corresponds to a dual curvelet localized near  $(t_0, \theta_0)$  in the Radon domain oriented with slope  $u_0$ . ( $\tau_0$  is the direction, in radians, corresponding to slope  $u_0$ .)

The Radon transform of  $f$  is given by

$$Rf(\theta, t) = \begin{cases} 2(1-t^2)^{1/2} & -1 \leq t \leq 1 \\ 0 & |t| > 1 \end{cases}.$$

Hence, in the  $(t, \theta)$  plane, the Radon transform is singular along  $t = \pm 1$ . This is not a step-discontinuity, however, but rather a more gentle kind of discontinuity; it is said to be of order  $1/2$  which says that the Radon transform locally behaves like a square root function. Suppose that we wish to reconstruct –from Radon data– the boundary of the disk in the vicinity of a point  $x_0$  such that  $|x_0| = 1$ . Curvelets located near  $x_0$  and aligned, with the edge, i.e. of codirection parallel to  $x_0$  such that  $(\cos \theta_0, \sin \theta_0) = \pm(x_{0,1}, x_{0,2})$  will then be especially useful (they will be associated with big coefficients). Applying (6) – (7) gives

$$t_0 = \pm 1, \quad \tau_0 = 0.$$

Hence, the corresponding dual elements are aligned with the edge in the Radon domain. This is the micro-local principle at work: in effect, the analysis detects edges at certain locations and orientations in the Radon domain and automatically synthesizes edges at corresponding locations and directions in the original domain.

The micro-local correspondence we just introduced is a powerful representation of the inverse problem. For instance, it gives a new understanding of limited angle tomography. In limited angle tomography, the range of angular values for which Radon data are available is restricted. For instance, it may be a subinterval of  $[0, 2\pi)$ . The micro-local correspondence then gives a very precise description of the kind of features one may hope to recover.

## 5. STATISTICAL ESTIMATION

### 5.1. Applications to Noisy Data

In the reproducing formula (5), the  $\kappa_s$  in (5) are tending to zero as  $s \rightarrow \infty$ , so the reproducing formula is very sensitive to the presence of nonzero terms at large values of  $s$ . In particular, it would be rather



foolish to use this formula *as is* on inaccurate data. For dealing with noisy data, we propose a rule of the general form

$$\hat{f} = \sum_{\mu} \delta([Y, U_{\mu}] \kappa_s^{-1}, t_s) \gamma_{\mu},$$

where  $\delta(\cdot, t)$  is a scalar thresholding nonlinearity with threshold  $t$ , and  $t_s$  are appropriate scale-dependent thresholds.

This makes sense: because the curvelet transform has its big coefficients at unpredictable locations (depending on the location of the edge curve), we cannot say a priori where the ‘important coefficients’ will be; therefore we apply thresholding.

## 5.2. Statistical Near-Optimality

In the introduction, we claimed that the method was amenable to rigorous proof. Suppose that the problem is to recover a function  $f$  from noisy Radon data. The object to be recovered is a function on  $R^2$  assumed smooth apart from a discontinuity along a  $C^2$  curve – i.e. an edge. In Ref. 14, we used a continuum white noise model and proved that the shrinkage can be tuned so that the estimator will attain, within logarithmic factors, the optimal estimation rate. In comparison, linear procedures – SVD included – obtain markedly suboptimal rates of convergence, as do ‘wavelet-like’ shrinkage methods.

## 6. CONCLUSION

In this paper, we showed how a new set of ideas could be applied for solving linear inverse problems. At this time however, we do not want to claim anything practical about our approach and leave the development of concrete applications for future work. Rather, our intention here was to establish our new methodology as a significant intellectual contribution – a contribution that gives some fresh insights about a problem that received an enormous degree of attention.

We simply want to express the belief that some day these theoretical results may translate into improved representations in practical tomography, although right now in dealing with real data we only have made experiments with small-scale problems. Success or failure in these experiments should not be taken as anything other than an indication of our software development skills and dedication.

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