Agenda

1. Polynomial nonnegativity
2. Sum of squares decomposition and SDP
3. Examples
4. Global optimization
5. Application
Polynomial nonnegativity

- Given a polynomial in $n$ variables $f(x_1, \ldots, x_n)$, does there exist $x \in \mathbb{R}^n$ such that $f(x) < 0$?

  If not, $f(x) \geq 0 \ \forall x \in \mathbb{R}^n$ (globally nonnegative)

- Problem is NP-hard and has many applications

- Example: sequence $a_1, \ldots, a_n \in \mathbb{N}$ can be partitioned if

  $$\sum_{i \in I} a_i = \sum_{i \in I^c} a_i, \quad I \subset \{1, \ldots, n\}$$

  i.e. if $p_{\text{min}} = 0$ where

  $$p_{\text{min}} = \inf p(x), \quad p(x) = \left(\sum_{i} a_i x_i\right)^2 + \sum_{i} (x_i^2 - 1)^2$$

  NP-complete problem
Certification

Does there exist \( x \in \mathbb{R}^n \) such that \( f(x) < 0 \)?

Answering yes is easy: find \( x \) such that \( f(x) < 0 \)

Answering no is hard: we need a certificate
Sum of squares decomposition

If there are polynomials s.t.

\[ f(x) = g_1^2(x) + \ldots + g_m^2(x) \]

then \( f \geq 0 \)

This is an easy checkable certificate called SOS decomposition

- How do we find the \( g_i \)'s?
- When does such a certificate exist?
**SOS and SDP**

\[ f: \text{ polynomial in } n \text{ variables of degree } 2d \]
\[ z: \text{ vector of all monomials of degree } \leq d \]

\[ f \text{ SOS } \iff \exists Q: Q \succeq 0 \text{ and } f = z^T Q z \]

- This is an SDP
- Number of components of \( z \) is \( \binom{n+d}{d} \)
\[ \text{If } f \text{ SOS, then the } g_i \text{'s are of degree } d \text{ at most since because of the squaring, there would be no cancellations in the highest powers (why?)} \]

\[ g_i = v_i^T z \implies f(x) = \sum_i [g_i(x)]^2 = \sum_i z^T v_i v_i^T z = z^T V V^T z \]

with \( V = [v_1, v_2, \ldots] \). Take \( Q = V V^T \)

\[ \text{If } Q \text{ is feasible, } Q = V V^T \quad V = [v_1, v_2, \ldots] \]

\[ f = z^T V V^T z = \sum_i (\langle v_i, z \rangle)^2 \]

number of squares = \( \text{rank}(Q) \)
Example

Can we write a polynomial as a quadratic form on monomials?

Is \( f(x, y) = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 2y^4 \) SOS?

\[
  f(x, y) = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \quad \quad Q = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}
\]

Equate coefficients:

\[
  \begin{align*}
  x^4 &= x^2 \cdot x^2 & \quad a = 1 \\
  x^3y &= x^2 \cdot xy & \quad 2 = 2b \\
  x^2y^2 &= xy \cdot xy &= x^2 \cdot y^2 & \quad 3 = d + 2c \\
  xy^3 &= xy \cdot y^2 & \quad 2 = 2e \\
  y^4 &= y^2 \cdot y^2 & \quad 2 = f
\end{align*}
\]

\[\implies f = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 - 2c & 1 \\ c & 1 & 2 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \text{ holds for all } c\]
Example, continued

If $Q(c) \succeq 0$ for some $c$, then $Q(c) = V^TV$.

$$Q(c) = \begin{bmatrix} 1 & 1 & c \\ 1 & 3 - 2c & 1 \\ c & 1 & 2 \end{bmatrix} \succeq 0 \iff -1 \leq c \leq 1$$

$c = -1 \implies Q = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}^T$

$$f = \left\| \begin{bmatrix} x^2 + xy - y^2 \\ y^2 + 2xy \end{bmatrix} \right\|^2 = (x^2 + xy - y^2)^2 + (y^2 + 2xy)^2$$

For $c = 0$, another SOS decomposition
Example

\[ f = 2x^4 + 2x^3y - x^2y^2 + 5y^4 \]

\[
= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T Q \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}
\]

\[ = q_{11}x^4 + 2q_{12}x^3y + (q_{22} + 2q_{13})x^2y^2 + 2q_{23}xy^3 + q_{33}y^4 \]

\[ f \text{ SOS} \iff \exists Q \succeq 0 \text{ s.t.} \]

\[ q_{11} = 2 \quad 2q_{12} = 2 \]

\[ 2q_{13} + q_{22} = -1 \quad 2q_{23} = 0 \]

\[ q_{33} = 5 \]
Which polynomials are SOS?

\[ f: \text{SOS} \iff f \geq 0 \text{ (globally positive)} \]

In one variable, converse is true

Which nonnegative polynomials are SOS?

- \( n = 1 \)
- \( d = 2 \)
- \( n = 2, d = 4 \)

Strict inclusion for all other \((n, d)\)
Hilbert’s 17th problem

Motzkin polynomial: \( f = x^4 y^2 + x^2 y^4 - 3x^2 y^2 + 1 \)

- \( f \geq 0 \) but not SOS
- However \((x^2 + y^2 + 1)f\) is SOS

Artin (1927) solved Hilbert’s 17th problem: for any polynomial \( f \),

\[
f \geq 0 \text{ on } \mathbb{R}^n \implies f = \sum_i \frac{f_i^2}{g_i^2} \quad f_i, g_i \text{ polynomials}
\]

→ every non-negative polynomial is a sum of squares of fractions

Interpretation: \( f \cdot q^2 \) is SOS for some \( q \)

Sometimes, shape of common denominator is known
Pólya & Reznick [1928 & 1995]: for any homogeneous polynomial $f$

$$f > 0 \text{ on } \mathbb{R}^n \setminus \{0\} \implies f \cdot \left( \sum_{i=1}^{n} x_i^2 \right)^r \text{ is SOS for some } r \geq 0$$

Example [Parrilo]

$$f = \sum_{i,j} Q_{ij} x_i^2 x_j^2, \quad Q = \begin{bmatrix}
1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 \\
\end{bmatrix}$$

$f$ is not SOS but $\left( \sum x_i^2 \right) f$ is SOS
Global optimization

\[ f^* = \inf_{\mathbb{R}^n} f(x) \]

Find largest \( \lambda \) s.t. \( f - \lambda \) is SOS

- Due to Shor
- This is an SDP
- If exact can recover optimal solution
- Surprisingly effective

\[ f^* = \sup_{\text{s.t. } f - \lambda \geq 0} \lambda \quad \text{relax} \quad \sup_{\text{s.t. } f - \lambda \text{ SOS}} \lambda \]

SDP \( \leq \) OPT

Example: \( f(x, y) = 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xy - 4y^2 + 4y^4 \)

\( \lambda = 1.0316 \) is exact!
Polynomial optimization

\[
\begin{align*}
\min & \quad f_0(x) \\
\text{s.t.} & \quad f_i(x) \geq 0 \quad i = 1, \ldots, m
\end{align*}
\]

- Feasible set \( C \)
- Quadratic module \( M(f) = \{s_0 + \sum_{i=1}^{m} s_i f_i : s_i \text{ SOS}\} \).
- Relaxation

\[
\inf_{x \in C} f_0 = \sup_{\lambda} \quad \text{s.t.} \quad f_0 - \lambda \geq 0 \text{ on } C
\]

\[
g \in M(f) \implies g \geq 0 \text{ on } C
\]

\[
f_0 - \lambda \geq 0 \text{ on } C \quad \text{(hard constraint)} \iff f_0 - \lambda \in M(f) \quad \text{(SOS but unbounded degree)} \iff f_0 - \lambda \in [M(f)]_{2t} \quad \text{(tractable SDP!)}
\]

\[
p \in [M(f)]_{2t} \iff p \in M(f) \& \deg(p) \leq 2t
\]
Sequence of relaxations

\[ \text{OPT(SDP}_t) \leq \text{OPT(SDP}_{t+1}) \leq \text{OPT(P)} \]

Relaxation by duality: duality, moment relaxation (Lasserre 2001)

Putinar (1993):

If \( \exists N \in \mathbb{N} \) s.t. \( N - \sum x_i^2 \in M(f) \) \( \quad [A] \)

Then \( p > 0 \) on \( C \) \( \implies \) \( p \in M(f) \)

If \( [A] \) holds, then \( \lim_{t \to \infty} \text{OPT(SDP}_t) = \text{OPT(P)} \)

See also Nie, Schwaighofer 2007
Stability analysis in control

To prove asymptotic stability of

\[ \frac{dx}{dt} = f(x) \quad (x \in \mathbb{R}^n) \]

it is sufficient to find a Lyapunov function

\[ V(x) > 0 \quad x \neq 0 \]
\[ \dot{V}(x) < 0 \quad x \neq 0 \]

- For linear systems: \( \dot{x} = Ax \), quadratic \( V(x) \):

\[ V(x) = x^T P x > 0 \quad \implies \quad P \succ 0 \]
\[ \dot{V}(x) = x^T (A^T P + PA)x < 0 \quad \implies \quad A^T P + PA \prec 0 \]

This is an SDP

- For nonlinear systems, SOS is a useful tool
Example: jet engine model with controller

\[
\begin{align*}
\dot{x} &= -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 \\
\dot{y} &= 3x - y
\end{align*}
\]

Try generic 4th-order polynomial Lyapunov function

\[
V(x, y) = \sum_{0 \leq j+k \leq 4} c_{jk} x^j y^k
\]

Find \( V \) obeying

- \( V \) SOS
- \(-\dot{V} \) SOS

This is an SDP!

Solution: \( V = 4.18x^2 - 1.58xy + 1.78y^2 - 0.13x^3 + 2.5x^2y = 0.34xy^2 + 0.61y^3 \\
+ 0.48x^4 - 0.05x^3y + 0.44x^2y^2 + 2 \cdot 10^{-6}xy^3 + 0.09y^4 \)
References


3. *Semidefinite Optimization and Convex Algebraic Geometry*, Edited by G. Blekherman, P. Parrilo and R. Thomas