1. (a) By the convolution theorem,
\[(\widehat{h * h^{-1}})(\omega) = \hat{h}(\omega)\hat{h^{-1}}(\omega),\]
And by construction,
\[(\widehat{h * h^{-1}})(\omega) = 1.\]
The two statements together give
\[\hat{h^{-1}}(\omega) = \frac{1}{\hat{h}(\omega)}.\]
(b) The statement actually holds as an “if and only if.” Indeed, when
\[h = a\delta_p \text{ for some } p \in \mathbb{Z} \text{ and nonzero scalar } a,\]
part (a) gives
\[\hat{h^{-1}}(\omega) = e^{ip} = \delta_{-p}(\omega),\]
from which it follows that \(\delta_p^{-1} = \delta_{-p}.\) Hence \(h^{-1} = a^{-1}\delta_{-p},\) which has compact support.

Now we turn to the actual problem: proving the converse. We assume both \(h\) and \(h^{-1}\) have finite support. Let \(n_{\text{max}}\) be the largest integer such that \(h[n_{\text{max}}] \neq 0.\) Similarly, let \(m_{\text{max}}\) be the largest integer such that \(h[m_{\text{max}}] \neq 0.\) Then by construction,
\[\delta_0[n] = (h * h^{-1})[n] = \sum_{m \in \mathbb{Z}} h^{-1}[m]h[n - m] = \sum_{m \leq m_{\text{max}}} h^{-1}[m]h[n - m] = \sum_{m = n - n_{\text{max}}}^{m_{\text{max}}} h^{-1}[m]h[n - m].\]
If \(n_{\text{max}} + m_{\text{max}} \neq 0,\) then taking \(n = n_{\text{max}} + m_{\text{max}}\) yields that
\[h^{-1}[m_{\text{max}}]h[n_{\text{max}}] = \delta_0[n_{\text{max}} + m_{\text{max}}] = 0.\]
This contradicts the optimality of either \(n_{\text{max}}\) or \(m_{\text{max}},\) forcing \(n_{\text{max}} + m_{\text{max}} = 0.\)

Next let \(n_{\text{min}}\) be the smallest integer such that \(h[n_{\text{min}}] \neq 0.\) Similarly, let \(m_{\text{min}}\) be the smallest integer such that \(h[m_{\text{min}}] \neq 0.\) In analogy to the calculations above,
\[(h * h^{-1})[n] = \sum_{m \in \mathbb{Z}} h^{-1}[m]h[n - m] = \sum_{m \geq m_{\text{min}}} h^{-1}[m]h[n - m] = \sum_{m = n - m_{\text{min}}}^{n - n_{\text{min}}} h^{-1}[m]h[n - m].\]
If \(n_{\text{min}} + m_{\text{min}} \neq 0,\) then the choice of \(n = n_{\text{min}} + m_{\text{min}}\) yields \(h^{-1}[m_{\text{min}}]h[n_{\text{min}}] = 0.\) Once again, this contradicts the optimality of either \(n_{\text{min}}\) or \(m_{\text{min}}.\) Consequently \(n_{\text{min}} + m_{\text{min}} = 0.\)
We can now write
\[ n_{\text{max}} + m_{\text{max}} = n_{\text{min}} + m_{\text{min}} = 0 \]
\[ \Rightarrow n_{\text{max}} - n_{\text{min}} = -(m_{\text{max}} - m_{\text{min}}), \]

But both \( n_{\text{max}} - n_{\text{min}} \) and \( m_{\text{max}} - m_{\text{min}} \) are nonnegative, and so we must have \( n_{\text{max}} = n_{\text{min}} \) and \( m_{\text{max}} = m_{\text{min}} \). Therefore, \( h \) (and \( h^{-1} \)) is supported on a single point. That is, \( h = a\delta_p \) for some \( p \in \mathbb{Z} \) and some \( a \in \mathbb{R} \).

2. **Note:** Here we use the notations introduced in Lectures 13 and 14.

(a) We begin with a simple experiment where spins move at constant velocity. Let \( \rho_f \) be positive and supported on a small disk of radius \( \delta > 0 \), let \( \mathbf{v} \) be a vector, and let \( \rho(x, t) = \rho_f(x - vt) \). Write down the signal equation for \( \rho \), and, using a suitable change of variables, write down the phase accrued by the spins on a reference frame where the spins are stationary. Which term contains information about \( \mathbf{v} \)?

Recall the signal equation from Lecture 14:
\[ s(t) = \int \rho(x, t) e^{-i\theta(x, t)} \, dx, \]
where \( \theta(x, t) \) is the phase accrued up to time \( t \), \( B(x, t) \) is the total magnetic field, and \( \omega(x, t) = \dot{\theta}(x, t) \) is the rate of precession. Here we assume
\[ B(x, t) = B_0 + G(x, t), \]
where \( B_0 \) is the static field in the \( z \)-direction, and \( G(x, t) = \langle g(t), x \rangle e_z \) is the gradient field with gradient \( g(t) \). If the spins are not moving, then
\[ \theta(x, t) = \gamma \int_0^t \| B(x, s) \| \, ds. \]

But now the spins are moving, namely at constant velocity \( \mathbf{v} \). Suppose a spin is at position \( x \) at time \( t \). Then at time \( s < t \), it was subjected to the magnetic field at position \( x - (t - s)v \). The phase accrued is then
\[ \theta(x, t) = \gamma \int_0^t \| B(x - (t - s)v, s) \| \, ds \]
\[ = \gamma t\| B_0 \| + \gamma \int_0^t \langle g(s), x - (t - s)v \rangle \, ds \]
\[ = \omega_0 t + \gamma \int_0^t \langle g(s), x \rangle \, ds + \gamma \int_0^t \langle g(s), \mathbf{v} \rangle \, ds. \]

In a moving reference frame, this is written
\[ \theta(x + tv, t) = \omega_0 t + \gamma \int_0^t \langle g(s), x + tv \rangle \, ds + \gamma \int_0^t (s - t) \langle g(s), \mathbf{v} \rangle \, ds. \]
\[ = \omega_0 t + \gamma \int_0^t \langle g(s), x \rangle \, ds + \gamma \int_0^t s \langle g(s), \mathbf{v} \rangle \, ds. \]
With a change of variables \( y = x - tv \), the signal equation becomes

\[
\begin{align*}
    s(t) &= \int \rho(x) e^{-i\theta(x,t)} \, dx \\
    &= \int \rho_f(y) e^{-i\theta(y+tv,t)} \, dy \\
    &= e^{-i\omega_0 t} e^{-i\gamma \int_0^t s(g(s), v) \, ds} \int \rho_f(y) e^{-i\gamma \int_0^t g(s) \cdot y} \, dy.
\end{align*}
\]

The term containing information about the velocity \( v \) is simply a modulation factor.

(b) The integrals that appear in part (a) for the phase accrued can be computed as

\[
\int_0^T \langle g(s), v \rangle \, ds = \int_0^{T/2} \langle g_0(s), y \rangle \, ds + \int_0^{T/2} \langle -g(s), y \rangle \, ds = 0,
\]

and

\[
\int_0^T s\langle g(s), v \rangle \, ds = \int_0^{T/2} s\langle g_0(s), v \rangle \, ds + \int_0^{T/2} s(s + T/2)\langle -g_0(s), v \rangle \, ds = -\frac{T}{2} \int_0^{T/2} \langle g_0(s), v \rangle \, ds,
\]

from where it follows the phase accrued after time \( T \) is

\[
\theta(x + tv, T) = \omega_0 T - \frac{\gamma T}{2} \int_0^{T/2} \langle g_0(s), v \rangle \, ds.
\]

Since each component of the gradient is assumed to be positive, the integral that appears above does not vanish, and it encodes the velocity information while canceling the effects of the nuclei.

(c) Using the calculations made in (b), the signal at time \( T \) is

\[
s(T) = e^{-i\omega_0 T} e^{i\gamma T \int_0^{T/2} \langle g_0(s), v \rangle \, ds} \int \rho_f(y) \, dy.
\]

If we write

\[
a_0 = \frac{\gamma T}{2} \int_0^{T/2} g_0(s) \, ds,
\]

then

\[
s(T) = e^{-i\omega_0 T} e^{i\langle a_0, v \rangle} \int \rho_f(y) \, dy.
\]

Since \( \omega_0 \) and \( T \) are known, there are only 3 unknowns. These are the total number of protons, and the two components of the velocity (we are working in \( \mathbb{R}^2 \) by assumption). A degrees-of-freedom argument shows we need a set of three different experiments. Note that we may have wrapping effects, and thus we need to make sure the inner product on the exponent is such that

\[
\langle a_0, v \rangle < \pi.
\]

Otherwise we would not be sure of the true value of \( \langle a_0, v \rangle \), since the phase \( \langle a_0, v \rangle + 2\pi \) produces the same signal as \( \langle a_0, v \rangle \). Note that our restriction (1) imposes constraints on the time \( T \) and the gradient field \( g_0 \) we can use to determine \( v \).
(d) We can perform three experiments to determine \( \mathbf{v} \). Consider the experiments represented by the vector \( \mathbf{a}_0 = 0 \) (by choosing \( \mathbf{g}_0 = 0 \)), and any \( \mathbf{a}_1, \mathbf{a}_2 \) linearly independent (by choosing two appropriate nonzero \( \mathbf{g}_0 \)). From (c) we deduce

\[
\frac{s_0(T)}{s_0(T)} = e^{-i\omega_0 T} \int \rho_f(y) \, dy, \quad \frac{s_1(T)}{s_0(T)} = e^{i\langle \mathbf{a}_1, \mathbf{v} \rangle}, \quad \frac{s_2(T)}{s_0(T)} = e^{i\langle \mathbf{a}_2, \mathbf{v} \rangle}.
\]

Clearly \( s_0(T) \) determines the total number of protons. For determining \( \mathbf{v} \), if we ignore wrapping effects, then

\[
\log \frac{s_1(T)}{s_0(T)} = i\langle \mathbf{a}_1, \mathbf{v} \rangle \quad \text{and} \quad \log \frac{s_2(T)}{s_0(T)} = i\langle \mathbf{a}_2, \mathbf{v} \rangle.
\]

We then have a linear system, whose solution determines \( \mathbf{v} \). Again, this method assumes there are no wrapping effects. Therefore we need, in addition to (II), that

\[
|\langle \mathbf{a}_1, \mathbf{v} \rangle| < \pi \quad \text{and} \quad |\langle \mathbf{a}_2, \mathbf{v} \rangle| < \pi,
\]

as otherwise we may get a solution modulo 2\( \pi \).

(e) We can use superposition in this case, as the phase accrued by the static spins is simply

\[
\theta(x, t) = \omega_0 t + \gamma \int_0^t \langle \mathbf{g}(s), \mathbf{x} \rangle \, ds,
\]

and the total associated signal is

\[
s(t) = e^{-i\omega_0 t} \int \rho_0(x) e^{-i\gamma \int_0^s \langle \mathbf{g}(s), \mathbf{x} \rangle \, ds} \, dx + e^{-i\omega_0 t} e^{-i\gamma \int_0^s \langle \mathbf{g}(s), \mathbf{v} \rangle \, ds} \int \rho_f(y) e^{-i\gamma \int_0^s \langle \mathbf{g}(s), y \rangle \, ds} \, dy.
\]

Let us use gradient fields having the same behavior as in (b). The signal at time \( T \) becomes

\[
s(T) = e^{-i\omega_0 T} \int \rho_0(x) \, dx + e^{-i\omega_0 T} e^{i\langle \mathbf{a}_0, \mathbf{v} \rangle} \int \rho_f(y) \, dy,
\]

where \( \mathbf{a}_0 \) was defined in (c). A degrees-of-freedom argument shows we now need 4 experiments to solve this system. Using \( \mathbf{a}_0 = 0 \) we obtain

\[
s_0(T) = e^{-i\omega_0 T} \int \rho_0(x) \, dx + e^{-i\omega_0 T} \int \rho_f(y) \, dy
\]

\[
= e^{-i\omega_0 T} \left( \int \rho_0(x) \, dx + \int \rho_f(y) \, dy \right).
\]

Next take \( \mathbf{a}_1, \mathbf{a}_2 \) and \( \mathbf{a}_3 \) to be distinct and non-collinear (notice this places conditions on \( T \) and the different choices of \( \mathbf{g}_0 \)), and observe the signal

\[
s_k(T) = e^{-i\omega_0 T} \left( \int \rho_0(x) \, dx + e^{i\langle \mathbf{a}_1, \mathbf{v} \rangle} \int \rho_f(y) \, dy \right).
\]

Denoting by \( \alpha \) the fraction of protons at rest, we have

\[
\frac{s_k(T)}{s_0(T)} = \alpha + (1 - \alpha) e^{i\langle \mathbf{a}_k, \mathbf{v} \rangle}, \quad k = 1, 2, 3.
\]

We can solve this system for each of the three unknowns. If, say, \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) are linearly independent, then the system

\[
\log \left( \frac{s_k(T)}{s_0(T)(1 - \alpha)} - \frac{\alpha}{1 - \alpha} \right) = i\langle \mathbf{a}_k, \mathbf{v} \rangle, \quad k = 1, 2,
\]
can be solved for \( \mathbf{v} \) in terms of \( \alpha \). The third equation in the original system then allows us to solve for \( \alpha \):

\[
\frac{s_3(T)}{s_0(T)} = \alpha + (1 - \alpha) e^{i(\mathbf{a}_3, \mathbf{v})}.
\]

Note here we again need to impose

\[ |\langle \mathbf{a}_k, \mathbf{v} \rangle| < \pi, \quad k = 1, 2, 3. \]