1. Let \( h^{-1} \) be the inverse of \( h \) defined by \( h * h^{-1}[n] = \delta[n] \), where \( \delta[0] = 1 \) and \( \delta[n] = 0 \) for \( n \in \mathbb{Z}, n \neq 0 \).

(a) Compute \( \hat{h}^{-1}(\omega) \) as a function of \( \hat{h}(\omega) \).

(b) Prove that if \( h \) has a finite support, then \( h^{-1} \) has finite support only if \( h[n] = \delta[n - p] \) for some \( p \in \mathbb{Z} \).

2. MR Flow Imaging. In lecture we discussed how Magnetic Resonance Imaging (MRI) is sensitive to motion. Here we will explore how this sensitivity can be used to analyze flow. The following is a very simple introduction to phase contrast techniques. For simplicity we will work in \( \mathbb{R}^2 \).

(a) We begin with a simple experiment where spins move at constant velocity. Let \( \rho_f \) be positive and supported on a small disk of radius \( \delta > 0 \), let \( \mathbf{v} \) be a vector, and let \( \rho(x, t) = \rho_f(x - \mathbf{v}t) \). Write down the signal equation for \( \rho \), and, using a suitable change of variables, write down the phase accrued by the spins on a reference frame where the spins are stationary. Which term contains information about \( \mathbf{v} \)?

(b) For \( T > 0 \) suppose we use the following function for the gradient fields

\[
g(t) = \begin{cases} 
g_0(t), & 0 < t \leq T/2, \\
-g_0(t - T/2), & T/2 < t \leq T, \\
0, & \text{otherwise},
\end{cases}
\]

for some smooth \( g_0 > 0 \) supported on \([0, T/2]\). What is the phase accrued by the spins in the reference frame used above?

(c) Using your results in (b) describe how many experiments you need to characterize \( \mathbf{v} \). In other words, how many different choices of \( g_0 \) yield a system of equations that you can solve for \( \mathbf{v} \) (and obtain a unique solution)? In particular, describe possible issues in determining \( \mathbf{v} \) for some choices of \( T \) and \( g_0 \). Note: Assume you can start all your experiments at \( t = 0 \).

(d) Using the expression in (a) and your result in (c) describe a method to compute \( \mathbf{v} \). Note: If you need to perform more than one measurement, assume you can start all of them at \( t = 0 \).

(e) In general we will have a combination of both a static set of spins, and spins that are moving. We assume these move at a constant velocity. Let \( \rho_0 \) be positive and supported on the unit disk. Define \( \rho(x, t) = \rho_0(x) + \rho_f(x - \mathbf{v}t) \). Describe how you can compute \( \mathbf{v} \) in this case. Hint: What can you say about the accrued phase in (b) if \( \mathbf{v} = 0 \)?