Problem 1

(a) By computing the eigenvalues and eigenvectors of $A^*A$, we have the SVD $A = UΣV^*$, where

$$U = \begin{bmatrix} \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad Σ = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{4}{\sqrt{10}} \\ \frac{4}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}.$$ 

Denote $U = [u_1, u_2]$ and $V = [v_1, v_2]$. The SVD is not unique. Actually, for each $i = 1, 2$, we can use the pair $(-u_i, -v_i)$ to replace $(u_i, v_i)$. Therefore, to find the SVD that has the minimal number of minus signs in $U$ and $V$, we have

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix}, \quad Σ = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}, \quad V = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{4}{\sqrt{10}} \\ \frac{4}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}.$$ 

(b) The first singular value $\sigma_1 = 10\sqrt{2}$, with the right singular vector $\begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{4}{\sqrt{10}} \end{bmatrix}$ and the left singular vector $\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{-\sqrt{2}}{\sqrt{2}} \end{bmatrix}$. The second singular value $\sigma_1 = 5\sqrt{2}$, with the right singular vector $\begin{bmatrix} \frac{4}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix}$ and the left singular vector $\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{-\sqrt{2}}{\sqrt{2}} \end{bmatrix}$.

(c) 
1. Let $A = [a_1, ..., a_n]$. Then $\|A\|_1 = \max_{1 \leq j \leq n} \|a_j\|_1 = \|a_2\|_1 = 16.$
2. By the SVD, we know $\|A\|_2 = \sigma_1 = 10\sqrt{2}$.
3. Let $A = \begin{bmatrix} a_1^* \\ \vdots \\ a_m^* \end{bmatrix}$. Then $\|A\|_\infty = \max_{1 \leq i \leq m} \|a_i^*\|_1 = \|a_2^*\|_1 = 15.$
4. By the SVD, we know $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} = 5\sqrt{10}$.

(d) Since $A = UΣV^*$, we have

$$A^{-1} = VΣ^{-1}U^* = \begin{bmatrix} 1/20 & -11/100 \\ 1/10 & -1/50 \end{bmatrix}$$

(e) The eigenvalues are the roots of the second order polynomial $\det(\lambda I - A)$, and we have the solution $\lambda_1 = \frac{3}{2} + \frac{\sqrt{391}}{2}i$ and $\lambda_2 = \frac{3}{2} - \frac{\sqrt{391}}{2}i$.

(f) By direct calculation, we have $\det A = (-2) \times 5 - 11 \times (-10) = 100$, so $\lambda_1 \lambda_2 = \det A$ and $\sigma_1 \sigma_2 = \|\det A\|.$

(g) The length of the ellipsoid’s major semi-axis is $\sigma_1 = 10\sqrt{2}$ and the length of the minor semi-axis is $\sigma_2 = 5\sqrt{2}$, so the area of the ellipsoid is $\pi \sigma_1 \sigma_2 = 100\pi.$
Problem 2

Suppose $\Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_m \end{bmatrix}$, $U = [u_1, \cdots, u_m]$ and $V = [v_1, \cdots, v_m]$. The SVD is $A = U\Sigma V^*$ and $A^* = V\Sigma U^*$, and we know that $Av_i = \sigma_i u_i$ and $A^* u_i = \sigma_i v_i$. Then

$$
\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} v_i \\ u_i \end{bmatrix} = \sigma_i \begin{bmatrix} v_i \\ u_i \end{bmatrix}
$$

In the same way, we have

$$
\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} v_i \\ -u_i \end{bmatrix} = -\sigma_i \begin{bmatrix} v_i \\ -u_i \end{bmatrix}
$$

Thus all the column vectors of $W = \begin{bmatrix} V & V \\ U & -U \end{bmatrix}$ are eigenvectors of $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$. Further

$$
WW^* = \begin{bmatrix} V & V \\ U & -U \end{bmatrix} \begin{bmatrix} V^* & U^* \\ V^* & -U^* \end{bmatrix} = 2I_{2m}
$$

Hence, $W$ is nonsingular and all the column vectors of $W$ form an eigenbasis of $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$ with corresponding eigenvalues $\sigma_1, \cdots, \sigma_m, -\sigma_1, \cdots, -\sigma_m$. Therefore, we have the eigenvalue decomposition

$$
\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = W \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} W^{-1}.
$$

Problem 3

The equation of a line is $y = a + bx$. We want to minimize

$$(1 - a)^2 + (-1 - a - b)^2 + (-2 - a - 2b)^2.$$  

In matrix form, we need to minimize

$$\|c - Az\|^2$$  

with $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $z = \begin{bmatrix} a \\ b \end{bmatrix}$

Solving the equation in the sense of least squares sense gives

$$z = (A^* A)^{-1} A^* c = \begin{bmatrix} 5 \\ -\frac{3}{2} \end{bmatrix}$$

Therefore, the linear equation is $y = \frac{5}{6} - \frac{3}{2}x$. 

Figure 1: Graphical representation of the singular vectors and values
Problem 4

(a) Set $A = U\Sigma V^*$, $x' = V^*x$, $b' = U^*b$. Using the fact that $U$ is unitary as well as $V^*$:

$$
\|b - Ax\|^2 + \lambda\|x\|^2 = \|UU^*b - U\Sigma V^*x\|^2 + \lambda\|V^*x\|^2
$$

We can rewrite this as

$$
\|b' - \Sigma x'\|^2 + \lambda\|x'\|^2 = \sum_j |b_j' - \sigma_j x_j'|^2 + \lambda|x_j'|^2.
$$

The problem completely decouples and we just need to minimize each term, namely, $|b_j' - \sigma_j x_j'|^2 + \lambda|x_j'|^2$. Taking derivaties, we see that the minimum is obtained by setting

$$
x_j' = w_j b_j', \quad w_j = \sigma_j/(\sigma_j^2 + \lambda).
$$

In summary

$$
x = VWU^*b, \quad W = \text{diag}(w_j).
$$

(b) Yes. We compute 1) $b' = U^*b$, 2) we multiply $b'$ by the diagonal matrix $W$, and 3) multiply the result with $V$.

(c) Yes we can so in some sense, there is no need to re-invent the wheel! By looking at the solution we have computed or by calculus, we can check that

$$
x = (A^*A + \lambda I)^{-1} A^*b \quad \Leftrightarrow \quad (A^*A + \lambda I)x = A^*b.
$$

If we replace $A$ by its SVD, we can check that this gives $\Box$. By calculus, we want to minimize

$$
\|Ax - b\|^2 + \lambda\|x\|^2 = \|x^* A^*Ax - 2x^* A^*b + b^*b + \lambda x^*x - x^*(A^*A + \lambda I)x + 2x^* A^*b + b^*b
$$

By setting the gradient to zero, we obtain

$$
2(A^*A + \lambda I)x - 2A^*b = 0
$$

which is $\Box$. So we need to solve the invertible system $\Box$. Since a least-square solver can solve linear systems, we can use it to solve our problem. If our solver $1a(A,b)$ takes as input $A$ and $b$, we would simply need to pass $A^*A + \lambda I$ and $A^*b$.

Problem 5

We first observe that

$$
\|Ax\|_{\infty} = \max_i \sum_j |a_{ij}x_j|
$$

and since

$$
|\sum_j a_{ij}x_j| \leq \sum_j |a_{ij}||x_j| \leq \sum_j |a_{ij}||x|_{\infty},
$$

we have that

$$
\|A\|_{\infty} \leq \max_i \sum_j |a_{ij}|.
$$

Now we can realize the equality. Assume without loss of generality that the first row is that with maximum row sum and choose $x_j = \bar{a}_{1j}/|a_{1j}|$ (this is the sign of $a_{1j}$ if everything is real valued). (If $a_{1j} = 0$, take $x_j = 0$.) Then $\|x\|_{\infty} = 1$ and

$$
(Ax)_1 = \sum_j |a_{1j}|^2/|a_{1j}| = \sum_j |a_{1j}| = \max_i \sum_j |a_{1j}|.
$$

This proves the claim.