Problem 1. Consider the vector space $\mathbb{R}^{n \times n}$ over $\mathbb{R}$, let $S$ denote the subspace of symmetric matrices, and let $\mathcal{R}$ denote the subspace of skew-symmetric matrices. For matrices $X, Y \in \mathbb{R}^{n \times n}$, define their inner product by $\langle X, Y \rangle = \text{Tr}(X^T Y)$. Show that, with respect to this inner product, $\mathcal{R} = S^\perp$.

Solution. We first remark that

$$\langle X, Y \rangle = \sum_{ij} X_{ij} Y_{ij} = \langle Y, X \rangle;$$

that is, the dot product is symmetric in $X$ and $Y$.

Now let $S$ be a symmetric matrix and $\mathcal{R}$ be skew symmetric. We have

$$\langle R, S \rangle = \text{Tr}(R^T S) = -\text{Tr}(RS) = -\text{Tr}(SR) = -\langle S, R \rangle = -\langle R, S \rangle.$$

The second to last equality uses $\text{Tr}(AB) = \text{Tr}(BA)$ while the last uses the fact that the dot product is symmetric. Hence, we conclude that

$$\langle R, S \rangle = -\langle R, S \rangle \iff \langle R, S \rangle = 0.$$

This proves that $\mathcal{R} \subset S^\perp$. Recall however that since any matrix $A \in \mathbb{R}^{n \times n}$ is the sum of a symmetric matrix and a skew symmetric matrix,

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2},$$

we have $S^\perp \subset \mathcal{R}$. Hence, $\mathcal{R} = S^\perp$. 

Problem 2. Let $V = \mathcal{P}^n = \{ p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n : \alpha_i \in \mathbb{R} \}$, and $W = \mathcal{P}^{n-1}$. Define $L : V \to W$ by $Lp = p'$, where $'$ denotes differentiation with respect to $x$. Is $L$ 1-1? Is $L$ onto?

Solution. $\forall p(x) \in V, p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n, \alpha_i \in \mathbb{R}$. By definition, $L(p(x)) = \alpha_1 + 2\alpha_2 x + \cdots + n\alpha_n x^{n-1}$.

Since $L(1) = 0, 1 \neq 0$, we have $1 \in \mathcal{N}(L)$, so $\mathcal{N}(L) \neq \{0\}$. Thus $L$ is not 1-1.

$\forall q(x) \in W, q(x) = \beta_0 + \beta_1 x + \cdots + \beta_{n-1} x^{n-1}, \beta_i \in \mathbb{R}$. Let

$$p(x) = \beta_0 x + \frac{1}{2} \beta_1 x^2 + \frac{1}{3} \beta_2 x^3 \cdots + \frac{1}{n} \beta_{n-1} x^n,$$

then $p(x) \in V$, and

$$L(p(x)) = \beta_0 + 2 \cdot \left( \frac{1}{2} \beta_1 \right) \cdot x + 3 \cdot \left( \frac{1}{3} \beta_2 \right) \cdot x^2 + \cdots + n \cdot \left( \frac{1}{n} \beta_{n-1} \right) \cdot x^{n-1} = q(x),$$

which shows $q(x) \in \mathcal{R}(L)$. Thus $W \subset \mathcal{R}(L)$. Since $L : V \to W$, $\mathcal{R}(L) \subset W$. Hence $\mathcal{R}(L) = W$, $L$ is onto.

In conclusion, $L$ is onto but not 1-1.
Problem 3. For $A \in \mathbb{R}^{m \times n}$, prove that $\mathcal{R}(A^+) = \mathcal{R}(A^T)$.

Solution. Let $T$ be the linear transformation in the definition of $A^+$ (Definition 4.1), i.e.

$$T : \mathcal{R}(A^T) \rightarrow \mathcal{R}(A), \quad x \mapsto Tx = Ax.$$  

We have seen in class that this transformation is invertible. If $T^{-1}$ denotes its inverse, then $\mathcal{R}(T^{-1}) = \mathcal{R}(A^T)$. It however follows from the definition of $A^+$ that $\mathcal{R}(A^+) = \mathcal{R}(T^{-1})$, which concludes the proof.
**Problem 4.** Find the Moore-Penrose pseudoinverse of $xy^T$ where $x, y \in \mathbb{R}^n$.

**Solution.** There are many ways of doing this. Set $A = xy^T$ and assume that $x \neq 0$ and $y \neq 0$ as otherwise $A = 0 \implies A^+ = 0$. Then $\mathcal{R}(A) = \text{span}(x)$ and $\mathcal{R}(A^T) = \text{span}(y)$. In other words, $A^+$ maps (i) $\text{span}(x)$ into $\text{span}(y)$ and (ii) $\text{span}(x)^\perp$ into 0. Therefore, $A^+$ is of the form

$$A^+ = \lambda yx^T$$

for some scalar $\lambda$. Indeed, a matrix as above obeys (i) and (ii) and this is the only way this can be done. It remains to find $\lambda$. By definition of the pseudo inverse, if $v \in \mathcal{R}(A^T)$, we must have $A^+ Av = v$. This means that we want

$$A^+ Ay = y.$$

A simple calculation shows that

$$A^+ Ay = \lambda \langle x, x \rangle \langle y, y \rangle y$$

so that

$$\lambda = \frac{1}{\|x\|^2 \|y\|^2}.$$

There is another solution based on the SVD since we can show that

$$A = \sigma_1 u_1 v_1$$

with $u_1 = x/\|x\|$, $v_1 = y/\|y\|$ and $\sigma_1 = \|x\| \|y\|$.
Problem 5.  (a) Let $X \in \mathbb{R}^{m \times n}$. If $X^T X = 0$, show that $X = 0$.

(b) Programming question.

Solution.  (a) Denote $X = [x_1 \ x_2 \ \cdots \ x_n]$, where $x_i \in \mathbb{R}^{m}$, $i = 1, 2, \cdots, n$.

From $X^T X = 0$ we know $\text{Tr}(X^T X) = 0$. On the other hand,

$$\text{Tr}(X^T X) = x_1^T x_1 + x_2^T x_2 + \cdots + x_n^T x_n.$$ 

Since $x_1^T x_1 \geq 0, x_2^T x_2 \geq 0, \cdots, x_n^T x_n \geq 0$, we know $x_1^T x_1 = 0, x_2^T x_2 = 0, \cdots, x_n^T x_n = 0$, which means $x_1 = 0, x_2 = 0, \cdots, x_n = 0$. Hence $X = 0$.

(b) The programming part can be found in another file.