Matrices are bold capital, vectors are bold lowercase and scalars or entries are not bold.

Problem 1
Since \(x^*y \in \mathbb{C}\) we have \((x^*y)^* = (\overline{x^*y})\) and so \(y^*x = (\overline{x^*y})\).
Applying this result give \(y^*Ax = y^*(Ax) = (Ax)^*y = x^*A^*y\).

Problem 2
Let \(v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\) and \(v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\). Now let us find an orthonormal basis of \(\text{span}(v_1, v_2)\). Let \(u_1 = v_1 - v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\)
and \(u_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\). Then \(u_1, u_2 \in \text{span}(v_1, v_2)\) and it is easy to check \(\|u_1\| = \|u_2\| = 1\) and \(u_1^*u_2 = 0\).
Since \(\text{dim}(\text{span}(v_1, v_2)) = 2\), we have \(\{u_1, u_2\}\) is an orthonormal basis of \(\text{span}(v_1, v_2)\). The orthogonal projection is then given by

\[
P = u_1u_1^* + u_2u_2^* = \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

Problem 3
Suppose \(\lambda\) is an eigenvalue of the unitary matrix \(U\), we claim that \(|\lambda| = 1\). Indeed, by the definition of eigenvalue, we know there exists a nonzero vector \(v\) such that \(Uv = \lambda v\), which implies \(\|Uv\| = |\lambda||v|\). Since \(U\) is unitary, we have \(\|Uv\| = \|v\|\). Therefore \(|\lambda||v| = \|v\|\). By the fact \(v \neq 0\), we have \(|\lambda| = 1\).

Problem 4
(a) Suppose \(\lambda\) is an eigenvalue of \(A\). Then there is a nonzero vector \(x \in \mathbb{C}^m\) obeying \(Ax = \lambda x\), and

\[
x^*Ax = x^*\lambda x = \lambda \|x\|^2. \tag{1}
\]

Now take the adjoint on both sides of this equation (note that this is a scalar equation so that this means that we are taking the complex conjugate). Since \(A^* = A\), the adjoint of the left side is \((x^*Ax)^* = x^*A^*x = x^*Ax\). The adjoint of right hand side is \(\overline{\lambda \|x\|^2}\) and, therefore,

\[
x^*Ax = \overline{\lambda \|x\|^2} \tag{2}
\]

By \((1)\) and \((2)\),

\[
\lambda \|x\|^2 = \overline{\lambda \|x\|^2},
\]
and since \(x \neq 0\), it must be that \(\lambda = \overline{\lambda}\). This implies that \(\lambda\) is a real number.

(b) Suppose \(\lambda_1\) and \(\lambda_2\) are two different eigenvalues of \(A\) corresponding to \(x\) and \(y\). Then \(Ax = \lambda_1 x\), \(Ay = \lambda_2 y\) and thus,

\[
y^*Ax = \lambda_1 y^*x, \quad x^*Ay = \lambda_2 x^*y. \tag{3}
\]

As in part (a), we take the adjoint of both sides of the first equation above. Since \(A^* = A\) and \(\lambda_1\) is real valued, we have

\[
x^*Ay = \lambda_1 x^*y.
\]
It follows that \(\lambda_1 y^*x = \lambda_2 y^*x\). Since \(\lambda_1 \neq \lambda_2\), it must be that \(y^*x = 0\); that is, \(x\) and \(y\) are orthogonal.
Problem 5

If $u = 0$ or $v = 0$, we know that $A = I$ and $A^{-1} = I$ ($\alpha$ can be any complex number).

Assume both $u$ and $v$ are nonzero vectors. We prove that $A = I + uv^*$ is nonsingular if and only if $(1 + v^*u)u \neq 0$. First assume $A$ is nonsingular. Then $Au \neq 0$, which implies $(1 + v^*u)u \neq 0$. Thus $1 + v^*u \neq 0$. Conversely, if $1 + v^*u \neq 0$, let $\alpha = -\frac{1}{1 + v^*u}$ and $B = I + \alpha uv^*$. Then

$$AB = (I + \alpha uv^*)(I + uv^*) = I + \alpha uv^* + uv^* + \alpha uv^*uv^* = I + (1 + \alpha v^*u)uv^* = I$$

Therefore $A$ is nonsingular and $A^{-1} = I + \alpha uv^*$ where $\alpha = -\frac{1}{1 + v^*u}$. This proves the claim.

If $A$ is singular, $u$ and $v$ are both nonzero vectors and $1 + v^*u = 0$. Suppose $x \in \text{null}(A)$. Then

$$(I + uv^*)x = x + (v^*x)u = 0.$$ 

Then $x \in \text{span}(u)$, which implies $\text{null}(A) \subseteq \text{span}(u)$. Conversely,

$$Au = (I + uv^*)u = (1 + v^*u)u = 0.$$ 

This implies $\text{null}(A) \supseteq \text{span}(u)$. Therefore $\text{null}(A) = \text{span}(u)$. 