Review on Random Vectors and Multivariate Normal Distribution

Mean and Covariance of Random Vectors

- We let \( Y = (Y_1, Y_2, \ldots, Y_n) \) be a random vector with mean \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \). In other words, in vector notations
  \[
  E(Y) = \mu.
  \]
- Introduce the covariance matrix \( \Sigma = \text{Cov}(Y) \) to be the \( n \times n \) matrix whose \((i, j)\) entry is defined by
  \[
  \Sigma_{ij} = \text{Cov}(Y_i, Y_j).
  \]
  where
  \[
  \text{Cov}(Y_i, Y_j) = E[(Y_i - E(Y_i))(Y_j - E(Y_j))].
  \]
- Let \( X = AY \) (\( A \) not random). The mean of the random vector \( X \) is given by
  \[
  E(X) = E(AY) = AE(Y) = A\mu,
  \]
  and the covariance is
  \[
  \text{Cov}(X) = A\text{Cov}(Y)A^T
  \]

The Multivariate Normal Distribution

- \( X \) is an \( n \)-dimensional random vector.
- \( X \) is said to have a multivariate normal distribution (with mean \( \mu \) and covariance \( \Sigma \)) if every linear combination of its component is normally distributed. We then write \( X \sim N(\mu, \Sigma) \).
  \[
  X \sim N(\mu, \Sigma) \iff a^T X \sim N(a^T \mu, a^T \Sigma a)
  \]
  - \( \mu \) is an \( n \times 1 \) vector, \( E(X) = \mu \)
  - \( \Sigma \) is an \( n \times n \) matrix, \( \Sigma = \text{Cov}(X) \).
- Density (\( \Sigma \) nonsingular)
  \[
  f(x) = \frac{1}{(2\pi)^{n/2}\left|\Sigma\right|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma (x - \mu)\right).
  \]
  Remark: if \( \Sigma \) is singular, \( X \) does not have a density.

Properties

Assume \( X \) is multivariate normal

1. Linear transformation. Suppose \( A \) is a non-random matrix, then
  \[
  AX \sim N(A\mu, A\Sigma A^T)
  \]
2. Marginal distribution. Set

\[ X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}, \quad X^{(1)} = (X_1, \ldots, X_p), X^{(2)} = (X_{p+1}, \ldots, X_n) \]

and

\[ \mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \]

Then

\[ X^{(1)} \sim N(\mu^{(1)}, \Sigma_{11}) \]

The marginal distribution of any subset of coordinates is multivariate normal.

3. The conditional distribution of \( X^{(2)} \) given \( X^{(1)} \) is multivariate normal. There is formula for the mean and covariance matrix.

4. Independence.

- \( X = (X_1, X_2) \) bivariate normal. \( X_1 \) and \( X_2 \) are independent if and only if they are uncorrelated.
- More generally, \( X = (X^{(1)}, X^{(2)}) \) multivariate normal. \( X^{(1)} \) and \( X^{(2)} \) are independent if and only if they are uncorrelated, i.e. \( \text{Cov}(X^{(1)}, X^{(2)}) = \Sigma_{12} = 0 \).