Combinatorics is the field of mathematics that deals with counting (try to think of words that sound like “combinatorics”). You might have learned a long time ago how to count sequentially, but there are lots of other things you might want to count, and sometimes it takes too long to count things one by one.

In the first problem set (Combinatorics Challenge # 1) you counted the number of hugs in a group of \(n\) people. This is the same value as the number of ways to pick 2 people from a collection of \(n\) people. But what if you want to pick 3 people from a collection of \(n\) people. Or more generally, what if you want to pick \(k\) people from a group of \(n\) people?

As you might guess, counting is pretty important in mathematics. As a result there are many different fields that have come out of combinatorics.

Before you start the problem set, read the Wikipedia articles on Factorial and Binomial Coefficient.

1) (5 points)
Click the links in this pdf to read the following articles on Wikipedia and then answer the questions. Combinatorics, Partition (Number Theory), Graph Theory, Order Theory, Probabilistic Combinatorics.

Did you read all of the articles? Which one of those topics was your favorite and why? (This will help me figure out what to put on the next challenge problems).

2) (5 points)
A permutation is an ordering of labelled objects. For example, if there are two people waiting in line to buy tickets at a movie, either person \(A\) is in front of person \(B\) or person \(B\) is in front of person \(A\). So we say that there are two permutations. If there are three people \(A, B, C\) waiting in line, there are 6 permutations: \(ABC, ACB, BAC, BCA, CAB, CBA\).

(a) How many permutations are there of four people waiting in line?
(b) How many permutations are there of five people waiting in line?
(c) How many permutations are there of nine people waiting in line?

After you finish this problem (or if you work on it for a really long time and get stuck) read the Wikipedia article about Permutations.

3) (5 points)
The symbol \(\binom{n}{k}\) is read as “\(n\) choose \(k\)”. It is the number of ways to choose \(k\) items from a set of \(n\) items in total. Ms. Young told me there was recently an election in your class where three students were chosen.

In a class of 25 students, how many different sets of 3 can you pick? Write this down as a single number, not an equation.

HINT: It will take you too long to list all the possibilities: there are more than 1000. You really need to read the two Wikipedia articles listed in this problem.
4) (5 points)
Read the link above about graph theory. When I say “graph” I will always mean a “simple undirected graph”. A graph $G(V,E)$ is a set of vertices $V$ and a set of edges $E$. In a graph of $n$ vertices, we label the vertices as $v_1, \ldots, v_n$. So for $i \in 1, \ldots, n$, $v_i \in V$ (in English “for each number $i$ in the set of numbers from 1 to $n$, vertex $v_i$ is an element of the set of vertices $V$”). We say that an edge $e_{i,j}$ is in the set of edges $E$ if vertex $v_i$ is connected to vertex $v_j$. There are no edges that connect a vertex to itself (i.e. $e_{i,i} \notin E$ or in English “the edge that connects vertex $v_i$ to itself is not an element of the set of edges $E$”). Each pair of vertices is connected by either 1 edge or 0 edges.

Here is an example of a graph with 6 vertices (labelled 1 through 6) and 7 edges $e_{1,3}, e_{1,6}, e_{2,3}, e_{2,5}, e_{3,5}, e_{5,6}, e_{6,8} \in E$.

(a) Draw a graph on 8 vertices (make sure you label them 1, 2, \ldots, 8) with edges $e_{1,2}, e_{1,3}, e_{1,6}, e_{2,3}, e_{2,7}, e_{4,8}, e_{5,6}, e_{5,7}, e_{6,8}, e_{7,8} \in E$

(b) A complete graph is a graph in which every possible edge exists in the graph.

How many edges are there in a complete graph on 10 vertices?

How many edges are there in a complete graph on 2000 vertices?

5) Fun Fact!
Any map that you can draw on a piece of paper can be colored with 4 colors such that no two regions that touch each other are colored the same. (This is useful for all you cartographers out there!) Although this result was conjectured in 1852, it was not proved to be true until 2004. But don’t worry! There are still plenty of really interesting problems left for you to answer! If you want, you can read more about the [Four Color Theorem] on Wikipedia.