310c: Homework Solutions 2017

Homework 1

Solution. 7.1.11

(a). This follows by a minor adaptation of the proof of Lemma 7.1.10. Specifically, let \( A_i \) denote the collection of \( x \in \mathbb{R}^i \) such that \( s \mapsto x(s) \) is linear, with \( A_i, i = 2, \ldots, 6 \) denoting the corresponding collections of all polynomial, constant, non-decreasing, differentiable, analytic functions, respectively. If \( A_i \in \mathcal{B}_i^i \) for some \( i \leq 6 \), then by Lemma 7.1.7 there exists a countable \( \{t_k\} \subset I \) such that the values of \( x(t_k) \) determine whether \( x(\cdot) \in A_i \) or not. However, for each \( 1 \leq i \leq 6 \), all \( x \in A_i \) and any \( u \in \mathbb{R} \), there exists \( u \notin A_i \) with \( x(s) = y(s) \) for all \( s \in \mathbb{R}, s \neq u \), so none of these six sets can be in \( \mathcal{B}_i^{\mathbb{R}} \). By the same reasoning, if either the collection \( C_i \) of all \( x \in \mathbb{R}^i \) that are continuous at some fixed \( t \), is in \( \mathcal{B}_i^{\mathbb{R}} \), or the (linear) subspace \( BV_1 \) of all elements of \( \mathbb{R}^i \) having finite total variation on \( \mathbb{R} \), is in \( \mathcal{B}_i^{\mathbb{R}} \), then there exists some countable \( \{t_k\} \subset \mathbb{R} \) such that \( y \in C_i \) (or \( y \in BV_1 \)), whenever \( x(t_k) = y(t_k) \) for some \( x \in C_i \) (or \( x \in BV_1 \), respectively), and all \( k \). This is of course false, for there always exists a monotone sequence \( k \mapsto u_k \in \mathbb{R} \), disjoint of \( \{t_k\} \), such that \( u_k \to t \) (hence \( y \in C_i \) requires also that \( u_k \to t \), and \( y \in BV_1 \) requires instead that \( \sum_k |y(u_{k+1}) - y(u_k)| \) is finite).

(b). The collections \( A_7 = \{x(\cdot) : x(s) = 0 \text{ for some } s \in \mathbb{R}\} \), \( A_8 = \{x(\cdot) : x(s) < x(t) \text{ for some } s < t \in \mathbb{R}\} \), are not in \( \mathcal{B}_i^\mathbb{R} \) for the same reason that \( A_i, i \leq 6 \) are not. Similarly, the set \( MX = \{x(\cdot) : x(s) \geq x(u) \text{ for some } s \in \mathbb{R}, u \in \mathbb{R} \) and all \( u \in (s-h, s+h)\} \) is not in \( \mathcal{B}_i^{\mathbb{R}} \) for the same reason that \( BV_1 \) is not.

(c). We apply the same reasoning as in part (a), namely suppose there exists \( A \subset C(\mathbb{R}) \) non-empty such that \( A \in \mathcal{B}_i^\mathbb{R} \). Then, there exists \( x \in A \) and a countable base \( \{t_k\} \subset \mathbb{R} \) for \( A \). In particular, any \( y \in \mathbb{R} \) such that \( y(t_k) = x(t_k) \) for all \( k \), must be in \( A \), hence also in \( C(\mathbb{R}) \). But, fixing any collection of values for \( y(\cdot) \) on \( \{t_k\} \) is not enough to guarantee its continuity throughout \( \mathbb{R} \), yielding a contradiction to our assumption that such \( A \in \mathcal{B}_i^\mathbb{R} \) exists.

In contrast, fixing a monotone sequence \( t_k \in \mathbb{R} \) whose limit is \( t_\infty \in \mathbb{R} \), some \( x_\infty \in \mathbb{R} \) and a sequence \( \{x_k\} \) whose limit as \( k \to \infty \) is not \( x_\infty \), the non-empty set \( A = \{x(\cdot) : x(t_k) = x_k, 1 \leq k \leq \infty \} \) is obviously in \( \mathcal{B}_i^\mathbb{R} \) and contains only functions \( x(\cdot) \) which are discontinuous at \( s = t_\infty \).

(d). We first show that the \( \sigma \)-algebra \( \mathcal{B}_i^\mathbb{R} \) contains no non-empty subset \( \Gamma \) of the set \( A = \mathcal{B}(\mathbb{R}) \) of all Borel measurable functions. Indeed, the existence of such \( \Gamma \) implies (as in part (c)), the existence of countable \( \{(t_k, x_k)\} \subset \mathbb{R} \times \mathbb{R} \) such that any \( y : \mathbb{R} \to \mathbb{R} \) for which \( y(t_k) = x_k, k = 1, 2, \ldots, \) must be in \( A \). However, this is not possible, since countable \( \{t_k\} \subset \mathbb{R} \) is a Borel measurable set, so there exists \( T \subset \mathbb{R} \setminus \{t_k\} \) not in the Borel \( \sigma \)-algebra \( \mathcal{B}_i^\mathbb{R} \) and hence \( y(s) = x1_T(s) + \sum_k x_k 1_{t_k}(s) \) is necessarily not in \( A \) (whenever \( x \notin \{0, x_k\} \)).
From the preceding we deduce that if \( A \in \mathcal{B}^\mathbb{R} \) then \( A \) must be a \((\mathcal{P}, \mathcal{B}^\mathbb{R})\)-null set, namely, \( A \subseteq G \in \mathcal{B}^\mathbb{R} \) for some \( G \) such that \( \mathcal{P}(G) = 0 \). It then follows that \( F = G^c \in \mathcal{B}^\mathbb{R} \) is a non-empty subset of \( A^c \). We claim that this can not be the case. Indeed, if such \( F \) exists, then by Lemma (4.3.4) there must exist countable \( \{ t_k, x_k \} \) such that there is no Borel function \( x(\cdot) \) on \( \mathbb{R} \) for which \( x(t_k) = x_k \) for all \( k \). In particular, setting \( x(s) = x \notin \{ x_k \} \) for all \( s \notin \{ t_k \} \), should result with a non-Borel measurable function. However, since \( \mathbb{R} \setminus \{ t_k \} \) is a Borel subset of \( \mathbb{R} \), for \( x(\cdot) \to \) to be non-Borel measurable would have required that some (countable) subset of \( \{ t_k \} \) is a non-Borel set, which is clearly impossible.

**Solution.** 7.1.12

Fixing \( h > 0 \) and \( t \geq 0 \), let \( Y = X_{t+h} - X_t \) and \( \mathcal{L} = \{ A \in \mathcal{F} : \mathcal{P}(A \cap C) = \mathcal{P}(A)\mathcal{P}(C) \text{ for all } C \in \mathcal{Y} \} \). Note that \( \mathcal{L} \) is a \( \pi \)-system. Also, for \( \mathcal{A} = (s_1, \ldots, s_m) \) such that \( 0 \leq s_1 < \cdots < s_m = t \) considering our assumption that \( X_{s_1}, X_{s_2} - X_{s_1}, \ldots, X_{s_m} - X_{s_m-1} \) are \( \mathcal{P} \)-mutually independent, with and without \( s_m+1 = t+h \), we deduce from Definition 1.4.3 that \( \{ X_{s_1} \in B_1 \} \cap \cdots \cap \{ X_{s_m} - X_{s_m-1} \in B_k \} \in \mathcal{L} \) for any \((B_1, B_2, \ldots, B_m) \in \mathcal{B}^m \). Thus, this \( \pi \)-system generates \( F_\mathcal{A} = \sigma(X_{s_1}, X_{s_2} - X_{s_1}, \ldots, X_{s_m} - X_{s_m-1}) \), by Dynkin's \( \pi \to \lambda \) theorem we have that \( F_\mathcal{A} \subseteq \mathcal{L} \).

The measurable map from \((X_{s_1}, X_{s_2}, \ldots, X_{s_m}) \to (X_{s_1}, X_{s_2} - X_{s_1}, \ldots, X_{s_m} - X_{s_m-1})\) is invertible, hence \( F_\mathcal{A} = \sigma(X_{s_1}, X_{s_2}, \ldots, X_{s_m}) \) (see Exercise 1.2.33). Let \( \mathcal{P} = \cup_{\mathcal{A}} \mathcal{F}_\mathcal{A} \) where the union is over all finite sets \( \mathcal{A} \subseteq [0, t] \). Since \( \mathcal{P} \) is a \( \pi \)-system, appealing once more to the \( \pi \to \lambda \) theorem you conclude that \( \mathcal{F}_\mathcal{T}^X = \sigma(\mathcal{P}) \subseteq \mathcal{L} \). That is, as claimed \( X_{t+h} - X_t \) is independent of \( \mathcal{F}_\mathcal{T}^X \).

**Solution.** 7.1.13

Given \( A_1, \ldots, A_n \in \mathbb{T} \), we define the corresponding f.d.d. as follows. First set \( B_{k1} = A_k = B_{k0}^\mathbb{R} \) for \( k = 1, \ldots, n \) and note that by monotonicity of \( A \to \mu(A) \), for each non-zero \( \mathbb{b} = (b_1, \ldots, b_n) \in \{0, 1\}^n \) the set \( A_\mathbb{b} = \bigcap_k B_{k\mathbb{b}} \mathbb{b} \in \{0, 1\}^n \) is in \( \mathbb{T} \). With \( N_\mathbb{b} \) a Poisson(\( \mu(A_\mathbb{b}) \)) random variable independent of \( (N_{\mathbb{b'}}, \mathbb{b'} \neq \mathbb{b}) \), let the random vector \((N_{A_1}, \ldots, N_{A_n}) \) have the same law as \((\sum_{\mathbb{b}=1}^{n} N_{A_\mathbb{b}}, k = 1, \ldots, n) \).

Then, considering \( n = 1 \) we see that \( N_{A_1} \) has the Poisson(\( \mu(A_1) \)) law for each \( A_1 \in \mathbb{T} \). Moreover, if \( A_k, k = 1, \ldots, n \) are disjoint sets then \( \sum_{k=1}^n b_k > 1 \) yields \( A_\mathbb{b} = 0 \) and hence \( N_{A_\mathbb{b}} = 0 \). So, in this case our construction results with mutually independent \( N_{A_\mathbb{b}}, k = 1, \ldots, n \). Similarly, for \( A_{n+1} = \cup_{k=1}^n A_k \) and disjoint \( A_k, k \leq n \), we have \( A_\mathbb{b} \) non-empty only when \( \mathbb{b} = 0 \) or when \( b_k = b_{n+1} = 1 \) is the only two non-zero coordinates of \( \mathbb{b} \). Hence, in this case \( N_{A_{n+1}} = \sum_{k=1}^{n+1} N_{A_k} \), as claimed.

Turning to check that these f.d.d. are consistent, fix \( n \) and a permutation \( \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \). Note that for index sets \( \{ A_{\pi(k)} \} \) the preceding constructs mutually independent \( N_{(b_{\pi(1)}, \ldots, b_{\pi(n)})} \) of Poisson(\( \mu(A_\mathbb{b}) \)) laws, hence the resulting law of \((N_{A_{\pi(1)}}, \ldots, N_{A_{\pi(n)}}) \) is merely the image of the law of \((N_{A_1}, \ldots, N_{A_n}) \) under the permutation \( \pi \) of the coordinates. That is, the identity (7.1.1) holds (where \( A_k \in \mathbb{T} \) serve as the index points \( t_k \) there). Next, fixing \( \{ A_k, k \leq n \} \) and non-zero \( \mathbb{b} \in \{0, 1\}^{n-1} \), since \( A_\mathbb{b} \) is the disjoint union of \( A_{\mathbb{b}0} \) and \( A_{\mathbb{b}1} \), we have by finite additivity of \( \mu(\cdot) \) that \( \mu(A_{\mathbb{b}1}) = \mu(A_{\mathbb{b}10}) + \mu(A_{\mathbb{b}11}) \). This applies for any non-zero \( \mathbb{b} \), so by the thinning property of Poisson variables we have the representation \( N_{\mathbb{b}1} = N_{\mathbb{b}0} + N_{\mathbb{b}1} \) (see Exercise 3.4.16 part (a)), and as a result the identity (7.1.2) also holds.

Finally, by Proposition 7.1.8 there exists a probability space \((\Omega, \mathcal{F}, \mathcal{P}) \) and on it a stochastic process \( A \to N_A \) indexed by \( \mathbb{T} \) which has the specified f.d.d.
Applying now the Kolmogorov-Centsov continuity theorem with $\alpha = 2$ and $\beta = p - 1$ yields the stated result.

(b). Clearly $G = X_t - X_s$ is a $\mathcal{N}(0, v)$ random variable and as in part (a)

$$v = \mathbb{E}[(X_t - X_s)^2] \leq 2C|t - s|^{p - 1},$$

for some finite constant $C$ and all $s, t \in \mathbb{I}$. Recall that if $G$ is a $\mathcal{N}(0, v)$ R.V. then $\mathbb{E}[G^{2k}] = c_k v^k$ for some universal constants $c_k$ (for example, from (1.3.18) one easily finds that $c_k = \frac{(2k)!}{2^k k!}$). Consequently, for some finite $C_k$

$$\mathbb{E}[(X_t - X_s)^{2k}] = c_k v^k \leq C_k |t - s|^{k(p - 1)}, \quad \forall t, s \in \mathbb{I}.$$

Applying now the Kolmogorov-Centsov continuity theorem with $\alpha = 2k$, $\beta = k(p - 1) - 1$ and $k = k(\gamma)$ large enough so $\gamma < (p - 1)/2 - 1/(2k)$, we deduce the existence of a locally $\gamma$-Hölder continuous modification of $\{X_t, t \in \mathbb{I}\}$. While this is not enough, by adapting the proof of the Kolmogorov-Centsov continuity theorem, one can actually construct one continuous modification of $\{X_t, t \in \mathbb{I}\}$ which is locally $\gamma$-Hölder for all $\gamma < (p - 1)/2$.

Solution. 7.2.13

(a). Clearly $\sup_{t \in \mathbb{J}} X_t \geq \sup_{s \in \mathbb{C} \cap \mathbb{J}} X_s$. Conversely, for any $t \in \mathbb{J}$ there exists a sequence $s_k \rightarrow t$, $s_k \in \mathbb{C}$ such that $X_t = \lim_k X_{s_k}$. Since $\mathbb{J}$ is open we have $\lim_k X_{s_k} \leq \sup_{s \in \mathbb{C} \cap \mathbb{J}} X_s$, so $\sup_{t \in \mathbb{J}} X_t = \sup_{s \in \mathbb{C} \cap \mathbb{J}} X_s$. The right-hand side is the supremum of countably many elements of $m_{\mathbb{F}X}$ so it belongs also to $m_{\mathbb{F}X}$.

(b). Clearly, $\sup_{t \in (s, s+h)} |X_t - X_s| = \sup_{t \in (s, s+h)} |X_t - X_s|$ and the claim follows by arguing as in (a) for the open interval $\mathbb{J} = (s, s+h)$.

Additional solutions provided.

Solution. 7.2.23

Since $\{X_t\}$ and $\{Y_t\}$ are modifications of each other, by definition $\mathbb{P}(N_t) = 0$ for $N_t = \{\omega : X_t(\omega) \neq Y_t(\omega)\}$ and each $t \geq 0$. Moreover, up to a null set $N_*$ such that $\mathbb{P}(N_*) = 0$ both $t \mapsto X_t(\omega)$ and $t \mapsto Y_t(\omega)$ are right-continuous functions. Therefore,

$$\bigcup_{t \geq 0} N_t \subseteq N_* \bigcup_{q \in \mathbb{Q}^+} N_q.$$

Assuming as usual that our probability space is complete, we consequently find that

$$\mathbb{P}(\exists t \geq 0, X_t \neq Y_t) \leq \mathbb{P}(N_*) + \sum_{q \in \mathbb{Q}^+} \mathbb{P}(N_q) = 0.$$

That is, $\{X_t\}$ and $\{Y_t\}$ are indistinguishable.

Solution. 7.2.24
(a). We use induction on \( k \). Starting with \( k = 1 \) we need only consider \( m = 0 \) and show that
\[
\sup_{t,s \in \mathbb{Q}^{(2,1)}_1 : |t-s| < 1} |x(t) - x(s)| \leq 2\Delta_{1,1}(x) = 2 \left[ |x(\frac{1}{2}) - x(0)| \vee |x(1) - x(\frac{1}{2})| \right].
\]
This inequality is trivial, for \( \mathbb{Q}^{(2,1)}_1 = \{0, \frac{1}{2}, 1\} \).
Next, assuming the stated inequality holds for \( k - 1 \) and any \( k - 1 > m \geq 0 \), we define for each \( s < t \in \mathbb{Q}^{(2,k)}_1 \)
\[
s' = \min\{u \in \mathbb{Q}^{(2,k-1)}_1 : u \geq s\}, \quad t' = \max\{u \in \mathbb{Q}^{(2,k-1)}_1 : u \leq t\}.
\]
Note that \( s', t' \in \mathbb{Q}^{(2,k-1)}_1 \) are such that \( s \leq s' \leq t' \leq t, s'-s \leq 2^{-k} \) and \( t-t' \leq 2^{-k} \).
Hence,
\[
|x(t) - x(t')| \leq \Delta_{k,1}(x), \quad |x(s') - x(s)| \leq \Delta_{k,1}(x).
\]
Further, if \( t-s < 2^{-m} \) for \( m = k - 1 \) then \( t' = s' \) so \( |x(t') - x(s')| = 0 \), whereas if \( t-s < 2^{-m} \) only for some \( m < k - 1 \), then by the induction hypothesis and the fact that \( t'-s' \leq t-s \),
\[
|x(t') - x(s')| \leq 2 \sum_{\ell=m+1}^{k-1} \Delta_{\ell,1}(x).
\]
Combining these inequalities we find that as claimed
\[
|x(t) - x(s)| \leq |x(t) - x(t')| + |x(t') - x(s')| + |x(s') - x(s)| \leq \Delta_{k,1}(x) + 2 \sum_{\ell=m+1}^{k-1} \Delta_{\ell,1}(x) + \Delta_{k,1}(x) = 2 \sum_{\ell=m+1}^{k} \Delta_{\ell,1}(x).
\]
(b). For any pair \( s, t \in \mathbb{Q}^{(2)}_1 \) satisfying \( 0 < |t-s| < 2^{-n} \) we have that \( 2^{-(m+1)} \leq |t-s| < 2^{-m} \) for some \( m \geq n \), and \( s, t \in \mathbb{Q}^{(2,k)}_1 \) for all \( k \) large enough. Thus, by part (a) and our assumption that \( \Delta_{\ell,1}(x) \leq 2^{-\gamma \ell} \),
\[
|x(t) - x(s)| \leq \sup_{u,v \in \mathbb{Q}^{(2,k)}_1, |u-v| < 2^{-m}} |x(u) - x(v)| \leq 2 \sum_{\ell=m+1}^{\infty} \Delta_{\ell,1}(x) \leq 2 \sum_{\ell=m+1}^{\infty} 2^{-\gamma \ell} = c_\gamma 2^{-\gamma (m+1)} \leq c_\gamma |t-s|^\gamma.
\]

**Solution. 7.2.9** Taking hereafter \( T = [0, \infty) \), recall that the topology induced on \( C(T) \) by uniform convergence on compact subsets of \( T \) is equivalent to that of the separable metric \( \mathcal{S} = (C(T), \rho(\cdot, \cdot)) \), where \( \rho(x, y) = \sum_j 2^{-j} \varphi(||x-y||_j) \) and \( \varphi(r) = r/(1+r) \). With \( C = \{A \subseteq C(T) : A \in \mathcal{B}\} \), upon following the proof of Lemma 7.2.8 we have that \( \mathcal{B}_0 = \mathcal{C} \) provided we show that any open ball \( B(x, r) = \{y \in C(T) : \rho(x, y) < r\} \) if \( \mathcal{S} \) is in \( \mathcal{C} \). To this end, recall that in the course of proving Lemma 7.2.8 we have shown that the sets \( B(x, r, m) = \{y \in C(T) : ||y-x||_m < r\} \) are in the \( \sigma \)-algebra \( \mathcal{C} \) for each \( x \in C(T), r > 0 \) and finite \( m \), hence it suffices to prove the following representation
\[
B(x, r) = \bigcup_{n \geq 1} \bigcup_{g \in \Gamma_n(r_n)} \bigcap_{j=1}^n B(x, q_j, j),
\]
in terms of countable unions/intersections of such sets, where \( r_n = r - 2^{-n} \) and for each positive integer \( m \),

\[
\Gamma_m(r) = \{ q \in \mathbb{Q}_+^m : \sum_{j=1}^{m} 2^{-j} \varphi(q_j) < r \}.
\]

Considering the sequence \( d_j = \|y - x\|_j \), convince yourself that this representation is a direct consequence of the identity

\[
\{ d \in \mathbb{R}_+^\infty : f_{\infty}(d) < r \} = \bigcup_{n \geq 1} \{ d \in \mathbb{R}_+^\infty : f_n(d) < r_n \},
\]

where \( f_{\infty}(d) = \sum_{j=1}^{\infty} 2^{-j} \varphi(d_j) \), \( f_n(d) = \inf \{ \sum_{j=1}^{n} 2^{-j} \varphi(q_j) : q_j > d_j, q_j \in \mathbb{Q}_+ \} \).

To verify this identity note that \( f_n(d) = \sum_{j=1}^{n} 2^{-j} \varphi(d_j) \) because of the continuity of \( \varphi : \mathbb{R}_+ \to [0, 1] \). As \( \sup d_j \sum_{j>n} 2^{-j} \varphi(d_j) \leq 2^{-n} \), if \( f_n(d) < r_n \) then necessarily \( f_{\infty}(d) < r \), whereas conversely \( f_{\infty}(d) < r \) implies that \( f_n(d) \leq f_{\infty}(d) < r_n \) for all \( n \) large enough.
Homework 2

Solution. (7.3.4) This exercise is stated and proved as [Dud89, Theorem 12.1.3]. (a). By Definition 7.3.1 if $X_t$ is a Gaussian process defined on an index set $\mathbb{T}$ then the joint distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is a multivariate normal $\mathcal{N}(\mu, \mathbf{V})$ for all $t_k \in \mathbb{T}, k = 1, 2, \ldots, n$ and $n < \infty$. By Proposition 8.5.14 such distribution is in turn uniquely determined by the vector $\mu = (m(t_1), m(t_2), \ldots, m(t_n))$ and the matrix $\mathbf{V} = (c(t_j, t_k))_{j,k=1,2,\ldots,n}$. Thus, the f.d.d. of such a process (which by Proposition 7.4.8 uniquely determine its law), are unambiguously defined by the corresponding mean function $m(\cdot)$ and auto-covariance function $c(\cdot, \cdot)$.

(b). As in part (a) the joint mean and auto-covariance functions specify the f.d.d. $\nu_{t_1,t_2,\ldots,t_n}$ to be $\mathcal{N}(\mu, \mathbf{V})$ which is well defined since the non-negative definiteness of the auto-covariance function, per condition (7.3.1), implies that the corresponding matrix $\mathbf{V}$ must also be non-negative definite. In view of our canonical construction (see Proposition 7.4.8), it thus suffices to show that these f.d.d. are consistent. To this end, first note that Definition 3.5.13 of $\nu_{t_1,t_2,\ldots,t_n} = \mathcal{N}(\mu, \mathbf{V})$ via its characteristic function is such that for any $B_k \in \mathcal{B}, k = 1, \ldots, n$ and permutation $\pi: \{1, \ldots, n\} \to \{1, \ldots, n\}$

$$
\nu_{t_1,t_2,\ldots,t_n}(B_1 \times B_2 \times \cdots \times B_n) = \nu_{\pi(1),\pi(2),\ldots,\pi(n)}(B_{\pi(1)} \times B_{\pi(2)} \times \cdots \times B_{\pi(n)}).
$$

That is, the consistency condition (7.4.11) holds. Next note that the multivariate normal characteristic functions $\Phi_{t_1,\ldots,t_n}(\cdot)$ corresponding to the f.d.d. $\nu_{t_1,t_2,\ldots,t_n}(\cdot)$ are such that for any $t_k \in \mathbb{T}$ and $\mathbf{q} \in \mathbb{R}^{n-1}$,

$$
\Phi_{t_1,\ldots,t_n-1}(\mathbf{q}) = \Phi_{t_1,\ldots,t_n}((\mathbf{q}, 0)).
$$

With $X \in \mathbb{R}^n$ and $X' \in \mathbb{R}^{n-1}$ denoting random vectors such that $\Phi_X(\cdot) = \Phi_{t_1,\ldots,t_n}(\cdot)$ and $\Phi_X'(\cdot) = \Phi_{t_1,\ldots,t_n-1}(\cdot)$, it is easy to see that such relation holds if $X'$ consists of the first $n-1$ coordinates of $X$ and as each characteristic function uniquely determines the law of the corresponding random vector, the f.d.d. $\nu_{t_1,t_2,\ldots,t_n}(\cdot)$ must satisfy also the consistency condition (7.4.12).

Solution. (7.3.10) (a). Note that for a weakly stationary, process $\{X_s, s \geq 0\}$ and any $t, h \geq 0$,

$$
0 \leq \text{Var}(X_{t+h} + X_t) = c(t+h, t+h) + c(t, t) + 2c(t, t+h) = 2[r(0) + r(h)].
$$

Consequently, $|r(h)| \leq r(0)$ for all $h \geq 0$. Further, if $r(h) = r(0)$ then by the preceding $\text{Var}(X_{t+h} - X_t) = 0$ for each $t \geq 0$, and consequently $X_{t+h} \overset{a.s.}{=} X_t$.

(b). When such process $\{X_s, s \geq 0\}$ has in addition independent increments, it follows that

$$
r(h) - r(0) = c(t, t+h) - c(t, t) = \text{Cov}(X_{t+h} - X_t, X_t) = 0.
$$

This applies for all $h \geq 0$, so by part (a) we deduce that $\{X_s, s \geq 0\}$ satisfies the condition (7.4.2) of Kolmogorov-Centsov theorem (with $\alpha = 2$, $c = 0$ and any $\beta > 0$). Hence, this S.P. has a continuous modification. Further, w.p.1., by part (a) also $X_h(\omega) = X_0(\omega)$ for all $h \in \mathbb{Q}$, $h \geq 0$. In particular, replacing $\{X_s\}$ by its continuous modification, the latter identity extends to all $h \geq 0$ and $\omega \in \Omega$, as claimed.

Solution. (7.3.13) (a). Fixing $n$ and $0 = s_0 \leq s_1 < \cdots < s_n$, let $a_j = \sqrt{s_j - s_{j-1}}, j \geq 1$ and $\{G_j\}$ be i.i.d. standard random variables. Then, $(B_{s_1}, \ldots, B_{s_n})$ have the distribution of
(\(S_1, \ldots, S_n\)), where \(S_0 = x\) and \(S_m = S_0 + \sum_{j=1}^{\infty} a_j G_j\). These are clearly multivariate normal distributions (as can be directly verified, or alternatively deduced from Exercise 3.5.20). Further, deleting a point \(s_k\) in this construction merely replaces in each of the sums \(S_m, m = k + 1, \ldots, n\) the term \(Y = a_k G_k + a_{k+1} G_{k+1}\) by \(Y' = a' G'\), where \(a' = \sqrt{s_{k+1} - s_k - 1}\) and \(G'\) is another standard normal variable independent of \(\{G_j\}\). Since \(a'^2 = a_k^2 + a_{k+1}^2\), it follows that \(Y \overset{D}{=} Y'\) (see Lemma 3.1.1), and the resulting joint law is thus not affected by this change. That is, the f.d.d.s satisfy the condition (7.1.3) and as such are consistent (see Lemma 7.1.4).

Considering the f.d.d. for \(s_1 = s, n = 1\), clearly \(E B_s = x\) and \(E (B_s - x)^2 = s\) for any \(s \geq 0\). Moreover, considering the f.d.d. for \(n = 2\) and \(0 \leq s_1 < s_2\), by the independence of \(B_{s_2} - B_{s_1}\) and \(B_{s_1} - x\)

\[
c(s_1, s_2) = \text{Cov}(B_{s_1}, B_{s_2}) = E[(B_{s_1} - x)(B_{s_2} - x)] = E[(B_{s_1} - x)^2] = s_1.
\]

Hence, \(c(s, t) = s \land t\), as required.

We note in passing that in view of Exercise 7.3.4 about the consistency of Gaussian f.d.d. one may replace the explicit construction provided here by a direct analytic proof that the function \(c(s, t) = s \land t\) is non-negative definite.

(b). From the representation we used in part (a) we have that for each \(0 \leq s_1 < s_2\) the R.V. \(B_{s_2} - B_{s_1}\) follows the \(\mathcal{N}(0, s_2 - s_1)\) law. Thus, in particular

\[
E[(B_t - B_s)^4] = 3(E[(B_t - B_s)^2])^2 = (t - s)^2
\]

(c.f. our solution for part (b) of Exercise 7.2.13), and setting \(I = [0, T]\), the existence of a continuous modification \(\{W_t, t \in I\}\) follows upon applying the Kolmogorov-Centsov continuity theorem (with \(\alpha = 4\) and \(\beta = 1\)).

(c). As summarized in Corollary 7.2.10 the continuous modification \(\{W_t, t \in I\}\) of the S.P. \(\{B_t, t \in I\}\) is a measurable mapping from \((\Omega, \mathcal{F})\) to the Borel \(\sigma\)-algebra of the separable metric space \((C(I), \| \cdot \|_\infty)\). Further, as in our solution for part (b) of Exercise 7.2.13 note that for \(k \geq 1\), some universal finite constants \(c_k\) and all \(t, s \geq 0\),

\[
E[(B_t - B_s)^{2k}] = c_k|t - s|^k.
\]

Therefore, fixing \(\gamma < 1/2\) and considering Kolmogorov-Centsov’s theorem with \(\alpha = 2k\) large enough yields the existence of a \(\gamma\)-locally Hölder continuous modification of \(\{B_t, t \in I\}\) (as in the solution of Exercise 7.2.13). Since any two continuous modifications of the same S.P. are indistinguishable, w.l.o.g. the same modification \(\{W_t, t \in I\}\) of part (b) is taken to be \(\gamma\)-locally Hölder continuous for all \(\gamma < 1/2\).

(d). With \(E B_t^2 = t\) depending on \(t\), this process is not even weakly stationary. However, by bi-linearity of the covariance,

\[
\text{Cov}(B_t - B_u, B_s) = c(t, s) - c(u, s) = t \land s - u \land s = 0
\]

for every \(0 \leq s \leq u < t\). As \(\{B_t, t \geq 0\}\) is a continuous time, Gaussian S.P., from Corollary 7.3.6 we have that \(\{B_t, t \geq 0\}\) has independent increments. Further, as shown in part (b), the R.V. \(B_t - B_s\) follows the \(\mathcal{N}(0, |t - s|)\) law, hence by Definition 7.3.11 the S.P. \(\{B_t, t \geq 0\}\) has stationary increments.

**Solution. 7.3.16**

(a). Suppose first that \(s > t\). Then \(W_s - W_t\) having the \(\mathcal{N}(0, s - t)\) law, is independent of \(W_t\). Therefore, by linearity of the expectation \(E[W_s | W_t] = W_t\) and \(\text{Var}[W_s | W_t] = \text{Var}[W_s - W_t | W_t] = E[(W_s - W_t)^2 | W_t] = s - t\).
Moving to deal with $s < t$, note that $(W_s, W_t)$ has the $\mathcal{N}(\mathbf{0}, \mathbf{V})$ law for the invertible two-dimensional covariance matrix $\mathbf{V} = \begin{bmatrix} s & s \\ s & t \end{bmatrix}$. Computing $\mathbf{V}^{-1}$ we thus arrive at the joint probability density function

$$f_{W_s, W_t}(x, y) = \exp\left(-\frac{x^2}{2s} - \frac{(y - x)^2}{2(t - s)}\right)/(2\pi \sqrt{s(t - s)}).$$

With the density of $W_t$ being $f_{W_t}(y) = \exp(-y^2/2t)/\sqrt{2\pi t}$, we find that the conditional density of $W_s$ given $W_t$ is $f_{W_s|W_t}(x|W_t)$, where

$$f_{W_s|W_t}(x|y) = \frac{f_{W_s, W_t}(x, y)}{f_{W_t}(y)} = \exp(-s^2y^2/2t)/\sqrt{2\pi s^2}$$

and $y = s(t - s)/t$. As the latter is the density of the $\mathcal{N}(st, v)$ law, we conclude that $E[W_s|W_t] = (s/t)W_t$ and $\text{Var}[W_s|W_t] = s(t-s)/t.$

(b) Since $\{W_t, t \geq 0\}$ is a MG of continuous sample functions, we know from Doob’s $L^2$ maximal inequality that for $I_n = [2^{n-1}, 2^n]$ and any $n \geq 1$,

$$E \left[ \sup_{s \in I_n} W_s^2 \right] \leq 4E[W_{2^n}^2] = 2^{n+2}.$$ 

Hence, for any $\varepsilon > 0$, by Markov’s inequality we deduce that

$$\varepsilon^2 P \left( \sup_{s \in I_n} |s^{-1}W_s| \geq \varepsilon \right) \leq E \left[ \sup_{s \in I_n} \left| s^{-1}W_s \right|^2 \right] \leq 2 - 2^{-(n-1)}E \left[ \sup_{s \in I_n} W_s^2 \right] \leq 2^{-n}.$$

Since these bounds are summable in $n$, by Borel-Cantelli I we have that for each $\varepsilon > 0$, w.p.1. $\sup_{s \in I_n} |s^{-1}W_s| < \varepsilon$ for all $n$ large enough. Considering $\varepsilon \downarrow 0$ we then conclude that $s^{-1}W_s \xrightarrow{a.s.} 0$ when $s \to \infty$.

(c) Considering $t \in \mathbb{I} = [0, 1]$, the f.d.d. of both $\tilde{B}_t = W_t - tW_1$ and $\tilde{B}_t = (1-t)W_{t/(1-t)}$ (with $\tilde{B}_1 = 0$), are laws of finite linear combinations of coordinates of certain Gaussian random vectors, hence in either case follow a multivariate normal distribution (see Exercise 3.5.20). Clearly, $E[\tilde{B}_t] = 0 = E[\tilde{B}_t]$ for all $t \in \mathbb{I}$. Thus, $\{B_t, t \in \mathbb{I}\}$ and $\{\tilde{B}_t, t \in \mathbb{I}\}$ are two Gaussian processes of zero mean functions, hence have the same law provided their auto-covariance functions coincide (see Exercise 7.3.3). Using bi-linearity of the covariance, we easily calculate these functions, as follows

$$\text{Cov}(\tilde{B}_s, \tilde{B}_t) = \text{Cov}((1-s)W_{s/(1-s)}, (1-t)W_{t/(1-t)})$$

$$= (1-s)(1-t)\left(\frac{s}{1-s} \wedge \frac{t}{1-t}\right) = s(1-t) \wedge t(1-s) = s \wedge t - st;$$

$$\text{Cov}(\tilde{B}_s, B_t) = \text{Cov}(W_s - sW_1, W_t - tW_1)$$

$$= \text{Cov}(W_s, W_t) - s \text{Cov}(W_1, W_t) - t \text{Cov}(W_1, W_s) + st \text{Cov}(W_1, W_1)$$

$$= s \wedge t - st - ts + st = s \wedge t - st.$$ 

In conclusion, as claimed $\{\tilde{B}_t, t \in \mathbb{I}\} \overset{D}{=} \{\tilde{B}_t, t \in \mathbb{I}\}$.

The continuity of $t \mapsto \tilde{B}_t$ on $[0, 1]$ follows from the sample path continuity of the Wiener process $\{W_s, s \geq 0\}$. Further, setting $s = \frac{1}{1+t}$ we see that $\tilde{B}_t = \frac{t}{1+s} \left(\frac{W_s}{s}\right)$, so from part (b) we deduce that $\tilde{B}_t \overset{a.s.}{\to} 0$ when $t \uparrow 1$. We thus conclude that w.p.1. the sample functions of $\{B_t, t \in \mathbb{I}\}$ are continuous.

(d) Let $\{B'_t, t \in \mathbb{I}\}$ denote the S.P. specified by the f.d.d. of $\{W_t, t \in \mathbb{I}\}$, conditioned upon $W_1 = 0$. The latter are conditional marginals of a multivariate normal distribution and hence themselves multivariate normal. Thus, $\{B'_t, t \in \mathbb{I}\}$ is a
Gaussian S.P. whose mean function is zero by part (a). So, as in part (c), it suffices to show that Cov\((B'_s, B'_t)\) = \(s \cap t - st\). The case \(s = t\) is covered by part (a), so without loss of generality assume hereafter that \(s < t\). Now, recall that \((W_s, W_t, W_1)\) follows a multivariate normal distribution of covariance matrix

\[
\begin{bmatrix}
s & s & s \\
\hline
s & t & t \\
\hline
s & t & 1
\end{bmatrix}.
\]

Since \(((W_s, W_t)|W_1 = 0)\) is the conditional marginal of this distribution on its first two coordinates, it has the covariance matrix

\[
\begin{bmatrix}
s & s \\
\hline
s & t
\end{bmatrix} - \begin{bmatrix}
s \\
1
\end{bmatrix}^{-1} \begin{bmatrix}
s & s \\
\hline
s & t
\end{bmatrix} = \begin{bmatrix}
s & s \\
\hline
s & t
\end{bmatrix} - \begin{bmatrix}
s^2 & st \\
\hline
st & t^2
\end{bmatrix}.
\]

We conclude that Cov\((B'_s, B'_t)\) = \(s - st\) for any \(s \leq t\), so \(\{B'_t, t \in \mathbb{I}\}\) has the same distribution as the standard Brownian bridge.

**Solution.**

(a). If \(\tau\) is a Markov time, then \(\{\tau \leq t - \epsilon\} \in \mathcal{F}_{(t-\epsilon)+} \subseteq \mathcal{F}_t\) for all \(t\) and \(\epsilon > 0\), hence \(\{\tau < t\} = \bigcup_n \{\tau \leq t - \frac{1}{n}\} \in \mathcal{F}_t\) for all \(t\). Conversely, if \(\{\tau < t\} \in \mathcal{F}_t\) for all \(t\), then \(\{\tau \leq t\} = \bigcup_n \{\tau < t + \frac{1}{n}\} \in \bigcap_n \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_t\), hence \(\tau\) is a Markov time.

(b). If \(\{\tau_n, n \in \mathbb{Z}_+\}\) are \(\mathcal{F}_t\)-stopping times, then \(\{\tau_1 + \tau_2 \leq t\} \subseteq \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathcal{F}_t\), so \(\tau_1 + \tau_2\) is an \(\mathcal{F}_t\)-stopping time, and \(\sup_n \tau_n \leq t\) = \(\bigcap_n \{\tau_n \leq t\} \in \mathcal{F}_t\), so \(\sup_n \tau_n\) is also an \(\mathcal{F}_t\)-stopping time.

Given \(A = \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\}\), the event \(\tau_1 + \tau_2 > t\) occurs if and only if the open interval \((t - \tau_1, \tau_2) \subseteq (0, t)\) is nonempty, thus contains some rational point \(q \in (0, t)\). Hence, \(A \cap \{\tau_1 + \tau_2 > t\} = A \cap B\), where

\[B = \bigcup_{q \in \mathbb{Q} \cap (0, t)} \{\tau_2 > q\} \cap \{\tau_1 > t - q\}.
\]

Since each set in the latter countable union belongs to \(\sigma(\mathcal{F}_q, \mathcal{F}_{t-q}) \subseteq \mathcal{F}_t\), it follows that \(A \cap \{\tau_1 + \tau_2 > t\} \in \mathcal{F}_t\). Further, for \(i = 1, 2\) we have that \(\{\tau_i > t\} \cap \{\tau_1 + \tau_2 > t\} = \{\tau_i > t\} \in \mathcal{F}_t\). Considering the union of these events we deduce that \(\{\tau_1 + \tau_2 > t\} \in \mathcal{F}_t\), hence \(\{\tau_1 + \tau_2 \leq t\} = \{\tau_1 + \tau_2 > t\}^c \in \mathcal{F}_t\). This applies for all \(t \geq 0\), so \(\tau_1 + \tau_2\) is an \(\mathcal{F}_t\)-stopping time.

(c). Since \(\mathcal{F}_t\)-Markov times are merely \(\mathcal{F}_{t+}\)-stopping times, in case \(\{\tau_n, n \in \mathbb{Z}_+\}\) are \(\mathcal{F}_t\)-Markov times, applying part (b) for the filtration \(\{\mathcal{F}_{t+}, t \geq 0\}\) we have that \(\tau_1 + \tau_2\) and \(\sup_n \tau_n\) are \(\mathcal{F}_t\)-Markov times. Moreover by part (a), \(\{\inf_n \tau_n < t\} = \bigcup_n \{\tau_n < t\}\) is in \(\mathcal{F}_t\) for all \(t \geq 0\), hence inf\(_n\) \(\tau_n\) is also an \(\mathcal{F}_t\)-Markov time. Finally,

\[
\lim \inf \tau_n = \sup \inf_k \tau_k \quad \& \quad \lim \sup \tau_n = \inf \sup_k \tau_k,
\]

hence these two random variables are also \(\mathcal{F}_t\)-Markov times.

(d). Suppose \(\tau_1\) and \(\tau_2\) are \(\mathcal{F}_t\)-Markov times. Fixing \(t \geq 0\), recall part (a) that \(A = \{\tau_1 < t\} \cap \{\tau_2 \leq t\} \subseteq \mathcal{F}_t\). Further, for any \(0 < q < t\) both \(\{\tau_2 > q\} \subseteq \mathcal{F}_q\) and \(\{\tau_1 > t - q\} \subseteq \mathcal{F}_{t-q}\), hence the set \(B\) of part (b) is in \(\mathcal{F}_t\) and by the same argument as in part (b), \(A \cap \{\tau_1 + \tau_2 > t\} = \hat{A} \cap B\) is also in \(\mathcal{F}_t\).

Now, if both \(\tau_1\) and \(\tau_2\) are strictly positive (for all \(\omega \in \Omega\), then \(\hat{A}^c = \{\tau_1 \geq t\} \cup \{\tau_2 \geq t\} \in \mathcal{F}_t\) is contained in \(\{\tau_1 + \tau_2 > t\}\). Consequently, in this case, for any \(t \geq 0\),

\[\{\tau_1 + \tau_2 > t\} = \hat{A}^c \cup (\hat{A} \cap B) \in \mathcal{F}_t.
\]
That is, $\tau_1 + \tau_2$ is an $F_t$-stopping time.

Alternatively, if $\tau_1$ is also an $F_t$-stopping time, then $\tilde{A} = \{\tau_1 \leq t\} \cap \{\tau_2 < t\} \in F_t$ and as in the preceding $\tilde{A} \cap \{\tau_1 + \tau_2 > t\} = \tilde{A} \cap B \in F_t$. Assuming in addition that $\tau_1 > 0$ for all $\omega \in \Omega$, results with $\tilde{A}^c = \{\tau_1 > t\} \cup \{\tau_2 \geq t\}$ which is contained in $\{\tau_1 + \tau_2 > t\}$. Hence, also in this case $\tau_1 + \tau_2$ is an $F_t$-stopping time.

Additional solutions provided.

**Solution. 7.3.9**

By definition, the weak stationarity of a (square-integrable) continuous time S.P. $X_t$ amounts to having constant mean function $m(t) = \mathbb{E}X_t$ and an auto-correlation function of the form $c(t, s) = r(|t - s|)$. In particular, for each $n$ and $t_1, \ldots, t_n$ both the vector $\mu = (m(t_1 + s), \ldots, m(t_n + s))$ and the matrix $V = (c(t_j + s, t_k + s))_{j,k=1,\ldots,n}$ are independent of $s \geq 0$. Assuming that $X_t$ is further a Gaussian process, from part (a) of Exercise 7.3.4 we have that the multivariate normal distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is also independent of $s \geq 0$. That is, the S.P. $X_t$ satisfies the equivalent formulation (7.3.2) of strict stationarity.

For a weakly stationary process which is not stationary, try $Z_t = (-1)^{N_t}Z_0$ where $N_t$ is a Poisson process of rate one which is independent of $Z_0$, a bounded variable such that $\mathbb{E}Z_0 = 0$ and $\mathbb{E}Z_0^3 = 1$. Indeed, by independence $m(t) = \mathbb{E}Z_t = \mathbb{E}(-1)^{N_t} \mathbb{E}Z_0 = 0$ and since $(-1)^2 = 1$, for any $t \geq s$,

$$c(t, s) = \mathbb{E}Z_tZ_s = \mathbb{E}Z_t^3\mathbb{E}(-1)^{N_t-N_s} = r(|t-s|),$$

because $N_t-N_s \overset{d}{=} N_{t-s}$ whose law depends only on $t-s$. Consequently, $Z_t$ is weakly stationary. However, with $N_t$ a Poisson$(t)$ variable, we have by independence that

$$\mathbb{E}Z_t^3 = \mathbb{E}Z_0^3\mathbb{E}(-1)^{3N_t} = \sum_{k=0}^{\infty} \frac{(-1)^3t^k}{k!} e^{-t} = e^{-2t}.$$ 

With $\mathbb{E}Z_t^3$ depending on $t$ the S.P. $Z_t$ can not be stationary.

Another example of this type, taken from [GS01] Example 8.2.5, is the weakly stationary $Y_t = A \cos(\pi t/2) + B \sin(\pi t/2)$, where $A, B$ are uncorrelated but not identically distributed variables of zero-mean and unit variance. This process can not be strictly stationary since the laws of $Y_0 = A$ and $Y_1 = B$ are not the same.

**Solution. 7.3.17**

(a). Using hereafter $\|g\|$ for $\|g\|_2$, clearly for any $t \geq s > 0$ and $x \in \mathbb{R}$,

$$g_t(s+x) = \frac{|t|^{H-1/2}}{\|g\|} g\left(\frac{s+x}{t}\right) = \frac{1}{\|g\|} \left(|t-s-x|^{H-1/2}\text{sgn}(t-s-x) + |s+x|^{H-1/2}\text{sgn}(s+x)\right),$$

Consequently, simple algebra yields that

$$g_t(s+x) - g_s(s+x) = \frac{1}{\|g\|} \left(|t-s-x|^{H-1/2}\text{sgn}(t-s-x) + |s|^{H-1/2}\text{sgn}(s)\right) = g_{t-s}(x).$$

The latter identity obviously applies also for $s = 0$ and implies that $\|g_t - g_s\| = \|g_{t-s}\|$ for all $t \geq s \geq 0$. Further, one easily checks that, from its definition,

$$\|g_t\|^2 = |t|^{2H}$$

for any $t \geq 0$. Therefore, for any $t \geq s \geq 0$,

$$2c(t,s) = \|g_t\|^2 + \|g_s\|^2 - \|g_{t-s}\|^2 = \|g_t\|^2 + \|g_s\|^2 - \|g_t - g_s\|^2 = 2 \int g_t(x)g_s(x)dx.$$ 

(b). In view of part (a) we have from part (b) of Exercise 7.3.3 that existence of fBM requires only the non-negative definiteness of $c(t,s) = \int g_t(x)g_s(x)dx$. This
non-negative definiteness is obvious, as for any \( a_k \in \mathbb{R}, t_k \geq 0, k = 1, \ldots, n \) and finite \( n \),
\[
\sum_{j=1}^{n} \sum_{k=1}^{n} a_j c(t_j, t_k) a_k = \int \left( \sum_{j=1}^{n} a_j g_{t_j}(x) \right)^2 dx \geq 0
\]
(compare with (7.3.1)).

Next, from the given expression for \( c(t, s) \) we deduce that the fBM \( \{X_t, t \geq 0\} \) of parameter \( H \) satisfies the identity
\[
E[(X_t - X_s)^2] = c(t, t) + c(s, s) - 2c(t, s) = |t - s|^{2H}.
\]
So, now follow the line of reasoning we have used in solving part (c) of Exercise 7.3.13. That is, as in our solution for part (b) of Exercise 7.2.13 since the continuous time S.P. \( X_t \) is Gaussian, we have that \( E[(X_t - X_s)^{2k}] = c_k |t - s|^{2Hk} \) for any \( k \geq 1 \), some universal finite constants \( c_k \) and all \( t, s \geq 0 \). Hence, fixing \( \gamma < H \) and considering Kolmogorov-Centsov’s theorem with \( \alpha = 2k \) large enough (and \( \beta = 2Hk - 1 \)), yields the existence of a \( \gamma \)-locally Hölder continuous modification of \( \{X_t, t \geq 0\} \) (and as in the solution of Exercise 7.2.13 the same modification can be taken to be \( \gamma \)-locally Hölder continuous for all \( 0 < \gamma < H \)).

(c). The auto-covariance function of fBM with parameter \( H = 1/2 \) is by definition \( c(t, s) = (t + s - |t - s|)/2 = t \wedge s \). The continuous modification of this S.P. is further centered, Gaussian and of continuous sample functions, hence it is a standard Wiener process (see Definition 7.3.12).

(d). Fixing non-random \( b > 0 \), since the fBM \( \{X_t, t \geq 0\} \) is a centered, Gaussian S.P. the same applies also for \( Y_t = b^{-H} X_{bt} \). Further, clearly,
\[
E(Y_t Y_s) = b^{-2H} c(bt, bs) = c(t, s)
\]
for any \( t, s \geq 0 \), namely the auto-covariance of \( \{Y_t, t \geq 0\} \) matches that of the fBM. We thus conclude that the two processes have the same law, as stated.

(e). In part (b) we saw that for any \( t \geq s \geq 0 \) the increment \( X_t - X_s \) of fBM is a Gaussian R.V. of zero-mean and variance \( |t - s|^{2H} \). In particular, its law depend only on \( t - s \) which by Definition 7.3.11 implies that the fBM is a S.P. of stationary increments (for any \( 0 < H < 1 \)).

Next, a Gaussian S.P. has independent increments if and only if its auto-covariance is of the form \( c(t, s) = h(t \wedge s) \) (c.f. the remark just after Corollary 7.3.6). In case of fBM this amounts to \( c(s + u, s) - c(s, s) = (s + u)^{2H} - u^{2H} \) independent of \( u \geq 0 \), which holds if and only if \( H = 1/2 \).
Solution. \textbf{S1.1.11}

(a). It is easy to verify that for any \(\sigma\)-algebras \(H \subseteq G\) and any \(H \in H\), the collection 
\[H^H = \{ A \in G : A \cap H \in H \}\]
is a \(\sigma\)-algebra (see part \(b\) of Exercise \textbf{1.1.13}). Since \(\tau\) is an \(F_t\)-stopping time, for any \(t \geq 0\) fixed, \(H_t = \{ \omega : \tau(\omega) \leq t \} \in F_t \subseteq F_\infty\), so 
\[(F_t)^{H_t}\]is a \(\sigma\)-algebra. Per Definition \textbf{S1.1.9} 
\[F_\tau = \bigcap_{t \geq 0} (F_t)^{H_t}\]must also be a \(\sigma\)-algebra.

Recall Definition \textbf{1.2.12} that \(\sigma(\tau) = \sigma(H_s, s \geq 0)\). Further, \(H_s \cap H_t = H_{s \wedge t} \subseteq F_{s \wedge t} \subseteq F_t\) for all \(s, t \geq 0\). As \(H_s \subseteq (F_t)^{H_t}\) for all \(t \geq 0\), clearly \(H_s \in F_\tau\) regardless of the value of \(s \geq 0\). With \(F_\tau\) a \(\sigma\)-algebra, it then follows that \(\sigma(\tau) \subseteq F_\tau\).

Suppose next that \(\tau(\omega) = t_0\) is non-random. Then, \(H_t = \Omega\) when \(t \geq t_0\) while \(H_t = \emptyset\) when \(t < t_0\). Since \(\mathcal{G} = F_\infty\) we have here 
\[(F_t)^{H_t} = \mathcal{F}_t\] and \((F_t)^0 = F_\infty\), it follows that \(F_\tau = \bigcap_{t \geq t_0} F_t = F_{t_0}\), as claimed.

(b). It is not hard to check that 
\[H^{H^H} = H^{H'} \cap H^{H}\]
for any pair \(H', H \in H\) (see part \(c\) of Exercise \textbf{1.1.13}). Setting \(H_t = \{ \omega : \theta(\omega) \leq t \}\) and \(\tilde{H}_t = \{ \omega : \theta(\omega) \wedge \tau(\omega) \leq t \}\), we clearly have \(\tilde{H}_t = \tilde{H}_t \cap H_t\), hence 
\[(F_t)^{H_t} = (F_t)^{H_t \wedge H_t}\]
for all \(t \geq 0\). Considering the intersections over \(t \geq 0\) we conclude that 
\[\mathcal{F}_{\theta \wedge \tau} = \mathcal{F}_{\theta} \cap \mathcal{F}_{\tau}\].

Next, note that \(\theta(\omega) < \tau(\omega)\) if and only if \(\theta(\omega) \leq q < \tau(\omega)\) for some \(q \in \mathbb{Q}^{(2)}\), and if in addition \(\theta(\omega) \leq q < \tau(\omega)\) then it suffices to consider \(q \in \mathbb{Q}^{(1)} := \mathbb{Q}^{(2)} \cup \{ t \}\). Hence, for any \(t \geq 0\),
\[B_t := \{ \omega : \theta(\omega) < \tau(\omega), \theta(\omega) \leq t \} = \bigcup_{q \in \mathbb{Q}^{(1)}} A_q,
\]
where \(A_q = \{ \omega : \theta(\omega) \leq s < \tau(\omega)\}\). Clearly, \(A_q \subseteq H_s \cap H_t \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t\) for any \(s \leq t\) and consequently \(B_t \subseteq \mathcal{F}_t\) for any \(t \geq 0\). Observing that \(B_t = \{ \theta < \tau \} \cap H_t\) we deduce that 
\[\{ \theta < \tau \} \subseteq \mathcal{F}_{\theta \wedge \tau}\].

Finally, by symmetry, \(\{ \theta \leq \tau \} = \{ \tau < \theta \}^c \subseteq \mathcal{F}_{\theta \wedge \tau}\) and therefore 
\[\{ \theta = \tau \} = \{ \theta \leq \tau \} \setminus \{ \theta < \tau \} \subseteq \mathcal{F}_{\theta \wedge \tau}\].

(c). From part \(b\) we know that \(\{ \theta \leq \tau \} \subseteq \mathcal{F}_{\theta \wedge \tau}\). So, if \(A \in \mathcal{F}_{\theta}\) then 
\[\tilde{A} = A \cap \{ \theta \leq \tau \} \subseteq \mathcal{F}_{\theta}\). Since 
\[\tilde{A} \cap H_t = A \cap H_t\]
we further deduce that \(\tilde{A} \cap H_t \subseteq \mathcal{F}_{\tau}\) for all \(t \geq 0\). That is, \(\tilde{A} \in \mathcal{F}_{\theta \wedge \tau}\). Consequently, fixing \(Z\) integrable we have by the definition of the C.E. \(W = E[Z|\mathcal{F}_{\theta \wedge \tau}]\) that
\[E[ZI_{\theta \leq \tau}IA] = E[ZI_{\tilde{A}}A] = E[W I_{\tilde{A}}] = E[W I_{\theta \leq \tau}IA].\]

This applies for any \(A \in \mathcal{F}_{\theta}\), so by definition \(WI_{\theta \leq \tau} \in m\mathcal{F}_{\theta}\) is merely the C.E. \(E[ZI_{\theta \leq \tau}|\mathcal{F}_{\theta}]\) and ‘taking out the known’ \(I_{\theta \leq \tau}\) we arrive at the stated identity
\[E[Z|\mathcal{F}_{\theta}]I_{\theta \leq \tau} = E[Z|\mathcal{F}_{\theta \wedge \tau}]I_{\theta \leq \tau}\]  \hspace{1cm} (*)

Turning to prove our second claim, consider the expected value of \((*)\) conditional on \(\mathcal{F}_{\tau}\). Since \(\mathcal{F}_{\theta \wedge \tau} \subseteq \mathcal{F}_{\tau}\), upon taking in what is known we deduce that
\[E[E(Z|\mathcal{F}_{\theta})|\mathcal{F}_{\tau}]I_{\theta \leq \tau} = E[E(Z|\mathcal{F}_{\theta})I_{\theta \leq \tau}|\mathcal{F}_{\tau}] = E[Z|\mathcal{F}_{\theta \wedge \tau}]I_{\theta \leq \tau}\].

Next, interchanging the roles of \(\theta\) and \(\tau\) in \((*)\) then replacing \(Z\) there by the integrable \(Z' = E(Z|\mathcal{F}_{\theta})\) we further have by the tower property that
\[E[Z'|\mathcal{F}_{\tau}]I_{\tau \leq \theta} = E[Z'|\mathcal{F}_{\theta \wedge \tau}]I_{\tau \leq \theta} = E[Z|\mathcal{F}_{\theta \wedge \tau}]I_{\tau \leq \theta}.
\]
Combining the two identities, we conclude that \(E[E(Z|\mathcal{F}_{\theta})|\mathcal{F}_{\tau}] - E[Z|\mathcal{F}_{\theta \wedge \tau}] = 0\) a.s. on the set \(\{ \theta \leq \tau \} \cup \{ \tau \leq \theta \} = \Omega\).
(d). Since \( \{\xi \leq t\} \in \mathcal{F}_t \) and \( \theta \leq \xi \) it follows that \( \{\xi \leq t\} = \{\xi \leq t\} \cap \{\theta \leq t\} \in \mathcal{F}_t \) for all \( t \geq 0 \). Hence, \( \xi \) is an \( \mathcal{F}_t \)-stopping time.

**Solution.**

(a). Set \( \langle X \rangle_t = E[X^2_t - \mathbf{E}X^2_0] \) and fix \( t > s \). Since the zero mean \( X_t - X_s \) is independent of \( \mathcal{F}_s^X \) it follows that \( E[X_s(X_t - X_s)|\mathcal{F}_s^X] = 0 \) a.s. hence also \( E[X_s(X_t - X_s)] = 0 \) and

\[
E[X_t^2 - \langle X \rangle_t|\mathcal{F}_s^X] = E[X_t^2 + 2X_s(X_t - X_s) + (X_t - X_s)^2|\mathcal{F}_s^X] - (\mathbf{E}X_t^2 - \mathbf{E}X_0^2)
\]

\[
= X_s^2 + E[X_t^2 - 2X_s(X_t - X_s) - X_s^2] - (\mathbf{E}X_t^2 - \mathbf{E}X_0^2)
\]

\[
= X_s^2 - \mathbf{E}X_s^2 + \mathbf{E}X_0^2 = X_s^2 - \langle X \rangle_s .
\]

Consequently, \( (X^2_t - \langle X \rangle_t, F^X_t) \) is a MG.

(b). Suppose \( \{X_t, t \geq 0\} \) is a martingale. Then, by definition \( E[X_t - X_s|\mathcal{F}_s^X] = 0 \) for any \( t \geq s \). In particular, \( E(X_t - X_s) = 0 \). Further, as a Gaussian process \( \{X_t, t \geq 0\} \) is square integrable, hence by the tower property and taking out what is known, for any \( u \leq s \)

\[
E[X_n(X_t - X_s)] = E[X_nE(X_t - X_s|\mathcal{F}_s^X)] = 0 .
\]

Fixing \( 0 \leq t_1 < \cdots < t_n < \infty \), it thus follows that the random vector \( \mathbf{Y} = (Y_1, \ldots, Y_n) \) composed of \( Y_1 = X_{t_1} \) and \( Y_k = X_{t_k} - X_{t_{k-1}} \), \( k \geq 2 \), has uncorrelated coordinates. Since \( \mathbf{Y} \) is a Gaussian random vector, by Definition \[3.5.13\] and Proposition \[3.5.14\] its characteristic function \( \Phi_{\mathbf{Y}}(\mathbf{\theta}) \) is merely the product of the corresponding functions \( \Phi_{Y_k}(\theta_k) \) for the coordinates of \( \mathbf{Y} \), which are thus mutually independent (see Exercise \[7.1.12\]). This in turn implies that \( \{X_t, t \geq 0\} \) has zero-mean independent increments (see Exercise \[7.1.12\]), so the conclusion of part (a) applies.

(c). By part (a) and our assumption that \( X_0 = 0 \) we have that \( (X_t^2 - \mathbf{E}X_t^2, t \geq 0) \) is a MG. Further, by stationarity of the zero-mean independent increments of \( \{X_t, t \geq 0\} \), for any \( t \geq 0 \) and \( n \geq 1 \),

\[
\mathbf{E}X_t^2 = E[(\sum_{k=1}^n X_{kt/n} - X_{(k-1)t/n})^2] = \sum_{k=1}^n \text{Var}(X_{kt/n} - X_{(k-1)t/n}) = n\mathbf{E}X_{t/n}^2 .
\]

Comparing this identity for \( t = q \in \mathbb{Q}^{(2,0)} \) and \( n = q2^\ell \) with the one for \( t = 1 \) and \( n = 2^\ell \), we obtain that \( \mathbf{E}X_{2^n}^2 = q\mathbf{E}X_{2^\ell}^2 \) for any \( q \in \mathbb{Q}^{(2)} \). By the monotonicity of \( t \mapsto \mathbf{E}X_t^2 \) (see Exercise \[3.2.3\]), upon considering dyadic rationals \( q_n \) and \( r_n \) such that \( q_n \uparrow s \in \mathbb{R}_+ \) and \( r_n \downarrow s \) we deduce that

\[
s\mathbf{E}X_{2^n}^2 = \sup_n \mathbf{E}X_{2^n}^2 \leq \mathbf{E}X_s^2 \leq \inf_n \mathbf{E}X_{2^n}^2 = s\mathbf{E}X_{2^n}^2 .
\]

Therefore, \( \mathbf{E}X_s^2 = s\mathbf{E}X_{2^n}^2 \) for all \( s \geq 0 \), which concludes the proof.

**Solution.**

(a). Adaptedness is clear, and the integrability of \( u_0(t, B_t, \theta) \) follows from the fact that \( B_t \) is \( \mathcal{N}(0, t) \) distributed, hence its moment generating function \( M_t(\theta) := E[\exp(\theta B_t)] = \exp(\theta^2 t/2) \) is finite for any \( \theta \in \mathbb{R} \). Since the Brownian motion has stationary independent increments, we also have

\[
E[u_0(t, B_t, \theta)|\mathcal{F}_s^B] = E[e^{\theta B_s + \theta(B_t - B_s) - \theta^2 t/2}|\mathcal{F}_s^B] = e^{\theta B_s - \theta^2 t/2} E[e^{\theta(B_t - B_s)}] = e^{\theta B_s - \theta^2 t/2} M_{t-s}(\theta) = e^{\theta B_s - \theta^2 t/2} e^{\theta^2 (t-s)/2} = u_0(s, B_s, \theta).
\]
Hence, $u_0(t, B_t, \theta)$ is an $\mathcal{F}_t^B$-martingale.

(b). With $u_0(t, y, \theta) = \exp(\theta y - \theta^2 y/2)$ let $a_{k,r}(\theta) = (-t/2)^r (y - \theta t)^{k-2r} u_0(t, y, \theta)$ and $c_{k,r} = k!/(k - 2r)! r!$. We shall prove the identity

$$u_k(t, y, \theta) = \frac{\partial^k}{\partial \theta^k} u_0(t, y, \theta) = \sum_{r=0}^{[k/2]} c_{k,r} a_{k,r}(\theta)$$

by induction on $k \in \mathbb{Z}_+$. Indeed, it trivially holds for $k = 0$ since $a_{0,0}(\theta) = u_0(t, y, \theta)$ and we have only to consider $r = 0$ in the sum. Next note that $\frac{\partial u_0}{\partial \theta} = (y - \theta t) u_0$, hence for $k \geq 1$,

$$\frac{d}{d\theta} a_{k-1,r} = (k - 1 - 2r)(-t) a_{k-2,r} + (y - \theta t) a_{k-1,r} = 2(k - 1 - 2r) a_{k-1,r+1} + a_{k,r}.$$

Moreover, if $r \leq m = \lfloor (k-1)/2 \rfloor$ then $c_{k-1,r} = (1 - 2r/k) c_{k,r}$ and for $s = r + 1$ we have that $(k - 1 - 2r)c_{k-1,r} = (s/k)c_{k,s}1_{r<(k-1)/2}$. Consequently, with $m' = \lceil k/2 \rceil$ we find that

$$\frac{d}{d\theta} \sum_{r=0}^{m} c_{k-1,r} a_{k-1,r} = 2 \sum_{r=0}^{m} (k - 1 - 2r)c_{k-1,r} a_{k-1,r+1} + \sum_{r=0}^{m} c_{k-1,r} a_{k,r}$$

$$= c_{k,m} a_{k,m}1_{m<(k-1)/2} + \sum_{s=1}^{m} 2s k^{-s} c_{k,s} a_{k,s} + \sum_{r=0}^{m} (1 - 2r/k) c_{k,r} a_{k,r}$$

$$= \sum_{r=0}^{m'} c_{k,r} a_{k,r}$$

since $m' = m$ unless $k$ is even in which case $m < (k-1)/2$ and $m' = m + 1 = k/2$.

In conclusion, if the stated identity holds for $k - 1 \geq 0$ then it holds also for $k$, since by the preceding

$$u_k(t, y, \theta) = \frac{\partial}{\partial \theta} u_{k-1}(t, y, \theta) = \frac{d}{d\theta} \sum_{r=0}^{m} c_{k-1,r} a_{k-1,r} = \sum_{r=0}^{[k/2]} c_{k,r} a_{k,r}(\theta).$$

By induction, this identity holds for all $k \in \mathbb{Z}_+$ and fixing $\theta = 0$ gives our claim that

$$u_k(t, y, 0) = \sum_{r=0}^{[k/2]} \frac{k!}{(k - 2r)! r!} (-t/2)^r y^{k-2r}.$$ 

(c). By definition $u_{k-1,h}(t, y, \theta) = h^{-1}(u_{k-1}(t, y, \theta + h) - u_{k-1}(t, y, \theta))$ converges to $u_k(t, y, \theta)$ when $h \to 0$ and by the mean-value theorem

$$\sup_{|\theta| \leq 1} |u_{k-1,h}(t, y, \theta)| \leq \sup_{|\eta - \theta| \leq 1} |u_k(t, y, \eta)| := U_k(t, y, \theta).$$

In the sequel we show that $\mathbf{E}U_k(t, B_t, \theta)$ is finite for any $t, \theta$ and $k$. It then follows by linearity and dominated convergence of C.E. that w.p.1. for all $\ell \geq 1$,

$$\mathbf{E}[u_\ell(t, B_t, \theta) | \mathcal{F}_s^B] = \lim_{h \to 0} h^{-1} \left( \mathbf{E}[u_{\ell-1}(t, B_t, \theta + h) | \mathcal{F}_s^B] - \mathbf{E}[u_{\ell-1}(t, B_t, \theta) | \mathcal{F}_s^B] \right)$$

$$= \left. \frac{\partial}{\partial \theta} \mathbf{E}[u_{\ell-1}(t, B_t, \theta) | \mathcal{F}_s^B] \right|_{\theta = 0}.$$

Iterating this identity for $\ell = k, k-1, \ldots, 1$, we deduce from part (a) that w.p.1.

$$\mathbf{E}[u_k(t, B_t, \theta) | \mathcal{F}_s^B] = \frac{\partial^k}{\partial \theta^k} \mathbf{E}[u_0(t, B_t, \theta) | \mathcal{F}_s^B] = \frac{\partial^k}{\partial \theta^k} u_0(s, B_s, \theta) = u_k(s, B_s, \theta).$$
Integrability follows from the same argument for exchanging the order of expectation and differentiation, so $u_k(t, B_t, \theta)$ is an $\mathcal{F}_t^B$-martingale.

As for the integrability of $U_k(t, B_t, \theta)$, first note that for all $k \geq 1$ and some $c_k$ finite,

$$|u_k(t, y, \theta)| \leq c_k (|y - \theta|^2 + t/2)^{k/2} u_0(t, y, \theta)$$

(for example, from the identity we derived in part (b) we easily see that this applies with $c_k = \max_x c_{k,x}$). Therefore, for some $C = C(k, t, |\theta|)$ finite,

$$U_k(t, y, \theta) \leq c_k (|y + (|\theta| + 1)t|^2 + t/2)^{k/2} e^{c|\theta|(|\theta| + 1)} \leq C e^{c|\theta|(|\theta| + 2)}$$

and this clearly yields the integrability of $U_k(t, B_t, \theta)$.

Finally, evaluating $c_{k,0} = 1$, $c_{k,1} = k(k-1)$, $c_{k,2} = k(k-1)(k-2)(k-3)/2$ and $c_{6,3} = 120$, the following are $\mathcal{F}_t^B$-martingales:

$$u_2(t, B_t, 0) = c_{2,0} B_t^2 + c_{2,1}(\frac{-t}{2}) = B_t^2 - t;$$
$$u_3(t, B_t, 0) = c_{3,0} B_t^3 + c_{3,1}(\frac{-t}{2}) B_t = B_t^3 - 3t B_t;$$
$$u_4(t, B_t, 0) = c_{4,0} B_t^4 + c_{4,1}(\frac{-t}{2}) B_t^2 + c_{4,2}(\frac{-t}{2})^2 = B_t^4 - 6t B_t^2 + 3t^2;$$
$$u_6(t, B_t, 0) = c_{6,0} B_t^6 + c_{6,1}(\frac{-t}{2}) B_t^4 + c_{6,2}(\frac{-t}{2})^2 B_t^2 + c_{6,3}(\frac{-t}{2})^3$$

$$= B_t^6 - 15t B_t^4 + 45t^2 B_t^2 - 15t^3.$$  

(d). For any $\theta \in \mathbb{R}$,

$$\left( \frac{\partial}{\partial t} + \frac{\partial^2}{2 \partial y^2} \right) u_0(t, y, \theta) = \frac{\partial^2}{\partial \theta^2} u_0(t, y, \theta) + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} u_0(t, y, \theta) = 0.$$  

In particular, since this expression is zero for all $\theta$, its $k$-th partial derivative with respect to $\theta$ is also zero, for any $k \geq 1$. The function $u_0(\cdot) \theta$ is the exponential of a trivariate polynomial in $(t, y, \theta)$, and as such it is real-analytic on $\mathbb{R}^3$. This means that its partial derivatives commute, hence

$$\left( \frac{\partial}{\partial t} + \frac{\partial^2}{2 \partial y^2} \right) u_k(t, y, \theta) = \frac{\partial^k}{\partial \theta^k} \left[ \left( \frac{\partial}{\partial t} + \frac{\partial^2}{2 \partial y^2} \right) u_0(t, y, \theta) \right] = 0.$$  

That is, $u_k(t, y, \theta)$ solves the heat equation, for any fixed $\theta \in \mathbb{R}$ and $k \geq 0$.

**Solution. 8.2.10**

(a). For $0 \leq s \leq t$ and $A \in \mathcal{F}_s$, by the martingale property of $(Z_t, \mathcal{F}_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ we have that

$$Q_s(A) = \mathbb{E}_{P_s}[Z_s 1_A] = \mathbb{E}[Z_s 1_A] = \mathbb{E}[Z_s 1_A] = \mathbb{E}_{P_s}[Z_s 1_A] = Q_t(A).$$

Since this applies for any $A \in \mathcal{F}_s$ we conclude that $Q_s = Q_t |_{\mathcal{F}_s}$.

(b). Fixing $0 \leq u \leq s \leq t$ and $Y \in L^1(\Omega, \mathcal{F}_s, \mathbb{Q}_s)$, for any $A \in \mathcal{F}_u$ we have by part (a) that

$$\mathbb{E}_{Q_s}[Y 1_A] = \mathbb{E}_{Q_s}[Y 1_A] = \mathbb{E}_{P_s}[Y 1_A Z_s] = \mathbb{E}[Y 1_A Z_s] = \mathbb{E}[I_A \mathbb{E}[Y Z_s | \mathcal{F}_u]]$$

$$= \mathbb{E}_{P_u}[I_A \mathbb{E}[Y Z_s | \mathcal{F}_u]] = \mathbb{E}_{Q_s}[I_A \mathbb{E}[Y Z_s | \mathcal{F}_u],$$

$$= \mathbb{E}_{Q_s}[I_A \mathbb{E}[Y Z_s | \mathcal{F}_u],$$
where the last equality is due to the fact that \( Z_u \in \mathcal{F}_u \) and the identity \( \mathbb{Q}_u = \mathbb{Q}_1 \mid \mathcal{F}_u \) of part (a). With the preceding holding for all \( A \in \mathcal{F}_u \), it follows by the definition of C.E. that \( \mathbb{Q}_t \)-a.s.

\[
\mathbb{E}_{\mathbb{Q}_t}[Y \mid \mathcal{F}_u] = \frac{\mathbb{E}[YZ_u \mid \mathcal{F}_u]}{Z_u}.
\]

The latter identity also holds \( \mathbb{P}_t \)-a.s. since the Radon-Nikodým derivative \( Z_t = d\mathbb{Q}_t/d\mathbb{P}_t \) is assumed to be strictly positive. Further, with \( \mathbb{P}_t = \mathbb{P} \mid \mathcal{F}_t \), the same applies \( \mathbb{P} \)-a.s.

(c). It is easily checked that the moment generating function of a Poisson(\( \lambda \)) R.V. \( X \) is \( m_\lambda(\theta) = \mathbb{E}[e^{\theta X}] = e^{\lambda(e^{\theta} - 1)} < \infty \) for any \( \theta \in \mathbb{R} \). In particular, the strictly positive process \((Z_t, t \geq 0)\) is integrable. Further, for any \( 0 \leq u \leq s, \) with \( N_s - N_u \) having the Poisson(\( \lambda(s - u) \)) law, independently of \( \mathcal{F}_u^N \), we find that for all \( \theta \in \mathbb{R} \),

\[
Z_u^{-1}\mathbb{E}[e^{\theta(N_s - N_u)Z_s \mid \mathcal{F}_u^N}] = e^{(\lambda - \lambda\theta)(s - u)}\mathbb{E}[(\lambda/\lambda)e^{\theta(N_s - N_u) \mid \mathcal{F}_u^N}]
\]

\[
= e^{(\lambda - \lambda\theta)(s - u)}m_{\lambda(s - u)}[\theta + \log(\lambda/\lambda)] = e^{\lambda(s - u)(e^{\theta} - 1)} = m_{\lambda(s - u)}(\theta).
\]

Noting that \( m_{\lambda(s - u)}(0) = 1 \) we have in particular that \( \mathbb{E}[Z_s \mid \mathcal{F}_u^N] = Z_u \), namely, that \((Z_t, \mathcal{F}_t)_{t \geq 0}\) is a strictly positive martingale, with \( \mathbb{E}[Z_0] = 1 \) (since \( N_0 = 0 \)). It then further follows from part (b) that for any \( 0 \leq u \leq s \leq T \) and all \( \theta \in \mathbb{R} \),

\[
\mathbb{E}_{\mathbb{Q}_T}[e^{\theta(N_s - N_u) \mid \mathcal{F}_u^N}] = Z_u^{-1}\mathbb{E}[e^{\theta(N_s - N_u)Z_s \mid \mathcal{F}_u^N}] = m_{\lambda(s - u)}(\theta).
\]

That is, under \( \mathbb{Q}_T \) the increment \( N_s - N_u \) follows the Poisson law of parameter \( \lambda(s - u) \) independently of \( \mathcal{F}_u^N \). Since this applies for any \( 0 \leq u \leq s \leq T \), we conclude that \((N_t, t \in [0, T])\) is a Poisson process of rate \( \lambda \) under the measure \( \mathbb{Q}_T \), for any \( T < \infty \).

**Solution. [8.221]**

Let \( \{X_t, t \geq 0\} \) be a uniformly integrable right-continuous sub-martingale with \( M_t = \sup_{0 \leq s \leq t} X_s \). Fixing \( y > 0 \), let \( A_t = \{M_t > y\} \) and recall [8.220] that

\[
\mathbb{P}(A_t) \leq y^{-1}\mathbb{E}[X_t I_{A_t}].
\]

Since U.I. implies \( L^1 \)-boundedness, Doob’s convergence theorem applies, so there exists \( X_\infty \in L^1 \) such that \( X_t \xrightarrow{a.s.} X_\infty \). Further, \( A_t \uparrow A_\infty \) hence \( X_t I_{A_t} \xrightarrow{a.s.} X_\infty I_{A_\infty} \), and by U.I. of \( \{X_t, t \geq 0\} \) this convergence also holds in \( L^1 \). So, taking \( t \to \infty \) in the preceding inequality we find that

\[
\mathbb{P}(M_\infty > y) = \mathbb{P}(A_\infty) \leq y^{-1}\mathbb{E}[X_\infty I_{A_\infty}] = y^{-1}\mathbb{E}[X_\infty I_{M_\infty > y}],
\]

and upon considering \( y \uparrow x > 0 \), we conclude that \( \mathbb{P}(M_\infty > x) \leq x^{-1}\mathbb{E}[X_\infty I_{M_\infty > x}] \).

Finally, it holds trivially that \( \mathbb{E}[X_\infty I_{M_\infty > x}] \leq \mathbb{E}[(X_\infty)_+] \).

**Additional solutions provided.**

**Solution. [8.1.12]**

(a). By definition \( A \in \mathcal{F}_\infty \) is in \( \mathcal{F}_\tau^+ \) if and only if \( A \cap \{\tau \leq s\} \in \mathcal{F}_{s+\epsilon} \) for all \( s \geq 0 \) and \( \epsilon > 0 \). Alternatively, setting \( t = s + \epsilon \) this amounts to

\[
A \cap \{\tau \leq t - \epsilon\} \in \mathcal{F}_t \quad \forall t, \epsilon > 0 \quad (*)
\]

We thus merely need to show that \((*)\) is equivalent to

\[
A \cap \{\tau < t\} \in \mathcal{F}_t \quad \forall t \geq 0 \quad (**). 
\]


Indeed, if (⋆) holds then with \( \{ \tau < t \} = \cup_{n>1} \{ \tau \leq t - 1/n \} \) and \( F_t \) being a \( \sigma \)-algebra, we get (⋆⋆). Conversely, assuming (⋆⋆) holds, with \( s \mapsto F_s \) non-decreasing we also have that \( A \cap \{ \tau < t - q \} \in F_t \) for all \( q > 0 \) and since
\[
\{ \tau \leq t - \epsilon \} = \bigcap_{q \in \mathbb{Q}, q < \epsilon} \{ \tau < t - q \},
\]
we get that (⋆) holds as well.

(b). Recall that \( \tau \leq \tau_1 \) and \( \tau < \tau_1 \) whenever \( \tau < \infty \). In particular, \( \{ \tau_1 \leq t \} \subseteq \{ \tau < t \} \) for each fixed non-random \( t \geq 0 \), and hence for any \( A \in F_{\tau_1} \), setting \( B := A \cap \{ \tau < t \} \in F_t \) we have that
\[
A \cap \{ \tau_1 \leq t \} = B \cap \{ \tau_1 \leq t \} \in F_t
\]
(since \( \tau_1 \) is an \( F_t \)-stopping time). This applies for all \( t \geq 0 \) and consequently, any such \( A \in F_{\tau_1} \) is also in \( F_{\tau_1} \), as claimed.

(c). Since \( \{ \tau < t \} = \bigcup_n \{ \tau_n < t \} \), if \( A \in \bigcap_n F_{\tau_n} \) then by part (a) for each non-random \( t \geq 0 \),
\[
A \cap \{ \tau < t \} = \bigcup_n [A \cap \{ \tau_n < t \}] \in F_t.
\]
By part (c) of Exercise 8.1.10 \( \tau \) is an \( F_t \)-Markov time and so again by part (a) we see that any such \( A \in \bigcap_n F_{\tau_n} \) is also in \( F_{\tau_+} \). Conversely, applying part (b) of Exercise 8.1.11 for the \( F_{\tau_1} \)-stopping times \( \tau \) and \( \theta = \tau_n \) (so \( \tau \wedge \theta = \tau \)), we deduce that \( F_{\tau_1} \subseteq F_{\tau_n} \) for all \( n \) and conclude that as claimed \( F_{\tau_+} = \bigcap_n F_{\tau_n} \).

If in addition \( \tau_n \) is an \( F_t \)-stopping time and \( \tau < \tau_n \) whenever \( \tau \) is finite, then from part (b) we further have that \( F_{\tau_1} \subseteq F_{\tau_n} \). When this applies for all \( n \), we thus deduce that \( F_{\tau_1} \subseteq \bigcap_n F_{\tau_n} \) and our claim that in this case \( F_{\tau_+} = \bigcap_n F_{\tau_n} \) is a direct consequence of the preceding identity.
Homework 4

Solution. [S.2.30]

(a). Fixing $u \geq 0$, recall Corollary [S.2.22] that $X_t^u := X_{t \wedge u}$ is a right-continuous $F_t$-submartingale. Further, by the submartingale property $\mathbb{E}[X_u | F_t] \geq X_t^u$ for all $t \geq 0$. Hence, $X_\infty^u = X_u \in mF_\infty$ is the last element of $\{X_t^u, t \geq 0\}$ (see Definition S.2.22). Therefore, applying Corollary 8.2.27 for this sub-MG and the $F_t$-stopping times $\tau \geq 0$ yields the stated claim that

$$\mathbb{E}[X_{u \wedge \tau} | F_\theta] = \mathbb{E}[X_\theta^u | F_\theta] \geq X_\theta^u = X_{u \wedge \theta},$$

(with equality in case of a MG).

(b). Recall Corollary [S.2.29] that $X_t := X_{t \wedge \tau}$ is a right-continuous $F_t$-submartingale. We assume here that $\{X_t^u, t \geq 0\}$ is U.I. so by Proposition [S.2.22] it has an integrable last element $X_\infty^u = \lim_t X_t^u$. By definition $X_\infty^u = X_\tau$ and applying Theorem [S.2.26] for the sub-MG $\{X_t^u, t \geq 0\}$ we further deduce that $X_\tau = X_\theta^u$ is integrable. Further, applying Corollary [S.2.27] for this sub-MG we see that

$$\mathbb{E}[X_\tau | F_\theta] = \mathbb{E}[X_\infty^u | F_\theta] \geq X_\theta^u = X_\theta.$$

Solution. [S.2.33]

Let $\tau = \inf\{t \geq 0 : Z_t \geq x\}$. Since the $F_t$-adapted process $Z_t$ has continuous sample path and $[x, \infty)$ is a closed set, it follows from Proposition [S.1.14] that $\tau$ is an $F_\tau$-stopping time. Consequently, $Z_\tau^u = Z_{t \wedge \tau}$ is a right-continuous $F_t$-MG (by Corollary [S.2.29]). Further, $0 \leq Z_\tau^u \leq x$ it is certainly U.I. and hence has a last element $Z_{\infty}^u = Z_\tau \in L^1$ (by Proposition [S.2.28] c.f. solution of part (b) of Exercise S.2.30). The sample path continuity of $Z_t$ and our assumption that $Z_0 = 1 < x$, $Z_t \to 0$ as $t \to \infty$ imply that $Z_{\infty}^u = x_{t \wedge \infty}$. Consequently, applying Doob's optional stopping theorem to the MG $\{Z_t^u, t \geq 0\}$ we find that

$$1 = \mathbb{E}[Z_0] = \mathbb{E}[Z_0^u] = \mathbb{E}[Z_{\infty}^u] = x \mathbb{P}(\tau < \infty) = x \mathbb{P}(\sup_{t \geq 0} Z_t \geq x).$$

Warning: To see that $Z_t \to 0$ is needed for our conclusion, simply consider the martingale $Z_t \equiv 1$ for which $\mathbb{P}(\sup_{t \geq 0} Z_t \geq x) = 0$ as soon as $x > 1$. Similarly, if $\mathbb{P}(Z_0 > 1) > 0$ then our formula for $\mathbb{P}(\sup_{t \geq 0} Z_t \geq x)$ has to be modified to $x^{-1} \mathbb{E}[Z_0 \wedge x]$.

Solution. [S.2.35]

(a). Fixing $b > 0$ and $r \in \mathbb{R}$ let $X_t^r := Z_t^r = W_t + rt$ and $\eta := \tau_b^r := \inf\{t \geq 0 : X_t \geq b\}$. Since $t \mapsto W_t(\omega)$ are continuous for all $\omega \in \Omega$, the same applies for $t \mapsto X_t(\omega)$. From Proposition [S.1.16] we thus deduce that $\eta = \tau_B$, the first hitting time of the closed set $B = [b, \infty)$ by the $F_t^W$-adapted S.P. $\{X_t, t \geq 0\}$, is indeed an $F_t^W$-stopping time, as claimed.

(b). Fixing $s > 0$ and $r \leq 0$ we note that $\theta = \theta(r, s) = \sqrt{r^2 + 2s} - r$ is merely the positive root of the quadratic equation $\frac{1}{4} \theta^2 + rt - s = 0$. Hence, by part (a) of Exercise [S.2.7]

$$Y_t = \exp(\theta X_t - st) = \exp(\theta W_t + (r\theta - s)t) = \exp(\theta W_t - \frac{\theta^2 t}{2})$$

is a non-negative $F_t^W$-martingale of continuous sample functions. Consequently, by Corollary [S.2.29] the same applies for the stopped process $V_t = Y_{t \wedge \eta}$ (for we have seen in part (a) that $\eta$ is an $F_t^W$-stopping time). Since $X_0 = 0 < b$ and $t \mapsto X_t$ is continuous, by the definition of $\eta$ necessarily $X_{t \wedge \eta} \leq b$ for all $t \geq 0$. Further,
if \( \eta(\omega) < \infty \) then \( X_n(\omega) = b \) and consequently \( Y_\eta = \exp(\theta b - s\eta) \). With \( \theta \geq 0 \) it follows that \( V_t < \exp(\theta X_t, \omega) \) \( \leq e^{\theta b} \) is uniformly bounded, hence a U.I. martingale. As such, we get from Doob’s optional sampling theorem that \( Y_\eta := \lim sup_{t \to \infty} V_t \) is integrable and \( EY_\eta = EY_0 = 1 \) (see part (b) of Exercise 8.2.30). Recall that \( s > 0 \), so if \( \eta(\omega) = \infty \) then \( V_t(\omega) \leq e^{\theta b - s t} \to 0 \) as \( t \to \infty \). Thus, the identity \( Y_\eta = \exp(\theta b - s\eta) \) extends to all \( \omega \in \Omega \), from which we conclude that

\[
1 = EY_\eta = E[e^{\theta b - s\eta}] = e^{\theta b} E[e^{-s\eta}].
\]

That is, \( E[e^{-s\tau_r^{(r)}]} = e^{-\theta(r,s)b} \) for all \( s, b > 0 \) and \( r \leq 0 \).

(c). Since \( \theta(r, s) \to \theta(r, 0) = -2r \) and \( e^{-s\eta} \uparrow I_{\eta < \infty} \) as \( s \downarrow 0 \), by monotone convergence we get from part (b) that \( P(\tau_b < \infty) = e^{2rb} \).

(d). Setting \( r = 0 \) we have from part (c) that \( P(\tau_b < \infty) = 1 \) for all \( b \), where \( \tau_b = \inf\{t \geq 0 : W_t \geq b\} \). Consequently, the event \( A = \bigcup_n \{\omega : \tau_n(\omega) < \infty\} \) occurs with probability one. Evidently, \( A \) is merely the event \( \lim sup_{t \to \infty} W_t = \infty \), which thus happens almost surely. Considering this result in case of the Brownian motion \( \forall t \in \mathbb{R} \), we further find that a.s. \( \lim inf_{t \to \infty} W_t = -\lim sup_{t \to \infty} W_t^{(1)} = -\infty \).

**Solution.** To simplify notations, fixing \( r \in \mathbb{R} \) and \( a, b > 0 \) we hereafter drop the superscript \( (r) \) subscripts \( a, b \).

(a). By part (d) of Exercise 8.2.35 w.p.1. as \( t \to \infty \) both \( \lim sup W_t^1 = \infty \) and \( \lim inf W_t = -\infty \). If \( r \leq 0 \), then \( \lim inf Z_t \leq \lim inf W_t = -\infty \), and if \( r \geq 0 \), then \( \lim sup Z_t \geq \lim sup W_t = \infty \). So, in either case, almost surely \( Z_t \) exits the interval \((-a, b)\) within a finite time \( \tau = \tau_{a,b}^{(r)} \). By the same argument as in proof of part (a) of Exercise 8.2.35 \( \tau \) is merely the first hitting time \( \tau_B \) for \( Z_t \) and the closed set \( B = (-\infty, -a] \cup [b, \infty) \), hence an \( F_t^W \)-stopping time.

From the definition of \( \tau \) and continuity of \( t \mapsto Z_t \) we deduce that \( Z_{t \wedge \tau} \subset [-a, b] \) is uniformly bounded. We thus have the uniformly bounded, hence U.I. stopped \( F_t^W \)-martingale \( Y_{t \wedge \tau} \) for \( Y_t = u_0(t, W_t, -2r) = \exp(-2r Z_t) \) (as in part (b) of Exercise 8.2.35 except for taking now \( s = 0 \) and \( \theta = -2r \)). Consequently, by Doob’s optional sampling theorem \( EY_\tau = EY_0 \) (for example, see part (b) of Exercise 8.2.30).

Further, if \( \tau \) is finite (which occurs a.s.), then \( Z_\tau \subset (-a, b) \) and \( Y_\tau \in \{e^{2ra}, e^{-2rb}\} \), respectively. So, setting \( A = \{Z_\tau = -a\}, B = \{Z_\tau = b\} \) and \( p_- = P(A) \) we have that

\[
1 = EY_\tau = EY_0 = p_- e^{2ra} + (1 - p_-) e^{-2rb},
\]

yielding the stated formula for \( p_- \) in case \( r \neq 0 \). Now, if \( r = 0 \) then \( Z_t = W_t \) is a martingale, so \( Z_{t \wedge \tau} \) is a U.I. martingale and by the optional stopping theorem

\[
0 = EZ_0 = EZ_\tau = -ap_- + b(1 - p_-),
\]

namely, \( p_- = b/(a + b) \) when \( r = 0 \), as claimed.

(b). Here \( r = 0 \) and for \( s \geq 0 \) and \( c = \pm 1 \) we get upon applying Doob’s optional stopping theorem for the MG \( Y_t = u_0(t, W_t, c\sqrt{2s}) = \exp(c\sqrt{2s} W_t - st) \) (such that \( Y_{t \wedge \tau} \) is uniformly bounded), that

\[
E[e^{c\sqrt{2s} W_t - st}] = EY_\tau = EY_0 = 1.
\]

Further, almost surely \( I_A + I_B = 1 \) and

\[
e^{c\sqrt{2s} W_t} = e^{-ac\sqrt{2s}} I_A + e^{bc\sqrt{2s}} I_B,
\]
resulting for \( c = \pm 1 \), with the identities

\[
e^{-a\sqrt{2}s}E[e^{-s^r}IA] + e^{b\sqrt{2}s}E[e^{-s^r}IB] = 1.
\]

We write these as a matrix equation

\[
\begin{bmatrix}
e^{-a\sqrt{2}s} & e^{b\sqrt{2}s} \\
e^{a\sqrt{2}s} & e^{-b\sqrt{2}s}
\end{bmatrix}
\begin{bmatrix}
E(e^{-s^r}IA) \\
E(e^{-s^r}IB)
\end{bmatrix}
= \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

The determinant of this two-dimensional linear system is \( \Delta = -2\sinh((a+b)\sqrt{2}s) \neq 0 \), so inverting the corresponding matrix we conclude that

\[
L_\tau(s) = EE^{-s^r} = \begin{bmatrix} 1, 1 \end{bmatrix}
\begin{bmatrix}
E(e^{-s^r}IA) \\
E(e^{-s^r}IB)
\end{bmatrix}
= \frac{1}{\Delta} \begin{bmatrix} e^{-b\sqrt{2}s} & -e^{b\sqrt{2}s} \\
e^{a\sqrt{2}s} & e^{-a\sqrt{2}s}
\end{bmatrix}
\begin{bmatrix} 1 \\ 1 \end{bmatrix}
= \frac{\sinh(a\sqrt{2}s) + \sinh(b\sqrt{2}s)}{\sinh((a+b)\sqrt{2}s)}.
\]

(c). We know from part (b) that \( L_\tau(s) = C(s)/D(s) \), where

\[
C(s) = \frac{\sinh(a\sqrt{2}s) + \sinh(b\sqrt{2}s)}{(a+b)\sqrt{2s}} = \sum_{k=0}^{\infty} c_k s^k,
\]

\[
D(s) = \frac{\sinh((a+b)\sqrt{2s})}{(a+b)\sqrt{2s}} = \sum_{k=0}^{\infty} d_k s^k,
\]

are analytic functions, with \( D(0) = C(0) = d_0 = c_0 = 1 \) and the positive coefficients \( c_k = 2^k (a^{2k+1} + b^{2k+1})/((a+b)(2k+1)!) \) and \( d_k = 2^k (a+b)2^k/(2k+1)! \). Thus, \( L_\tau(s) \) is infinitely differentiable at \( s = 0 \) and admits a converging power-series expansion

\[
L_\tau(s) = M(s),\end{equation}

where \( M(s) = \sum_{k=0}^{\infty} m_k s^k \) for \( m_0 = 1 \), some finite \( m_k, k \geq 1 \) and all \( s \in \mathbb{R}_+ \). Recall part (b) of Exercise 8.3.40 that \( E\tau^k = (-1)^k k! m_k \), so your goal here is merely to verify that \( m_1 = -ab \) and \( m_2 = ab(a^2 + 3ab + b^2)/6 \). The shortest way to accomplish this task is to compare the coefficients of \( s \) and of \( s^2 \) on both sides of the power-series identity \( C(s) = D(s)M(s) \). With \( d_0 = c_0 = m_0 = 1 \), doing so yields the linear equations \( c_1 = d_1 + m_1 \) and \( c_2 = m_1 d_1 + d_2 + m_2 \). Now, as \( d_1 = (a+b)^2/3 \) and \( c_1 = (a^2 - ab + b^2)/3 \), our first equation results with \( m_1 = c_1 - d_1 = -ab \). Then, as \( d_2 = (a+b)^4/30 \) and \( c_2 = (a^4 - a^3b + a^2b^2 - ab^3 + b^4)/30 \), our second equation results with

\[
m_2 = abd_1 + c_2 - d_2 = \frac{1}{3} ab(a+b)^2 - \frac{1}{6} ab(a^2 + ab + b^2) = \frac{ab}{6} (a^2 + 3ab + b^2),
\]

and so we are done (much faster than by the alternative of directly computing the first two derivatives of \( L_\tau(s) \) at \( s = 0 \)).

**Solution.** 8.3.41

(a). Recall that \( \mathcal{F}_t^Y = \sigma(Y_s : s \leq t) \) is generated by sets of form

\[
\{Y_{t_1} \in \tilde{A}_1, \ldots, Y_{t_n} \in \tilde{A}_n \}, \quad \tilde{A}_1 \in \tilde{S}, \quad 0 \leq t_1 < \ldots < t_n \leq t.
\]

Each mapping \( \Phi_s \) is invertible so such set can be re-expressed as

\[
\{X_{u(t_1)} \in A_1, \ldots, X_{u(t_n)} \in A_n \},
\]

where \( A_i = \Phi_{t_i}^{-1}\tilde{A}_i \in \mathcal{S} \) since \( \Phi_{t_i} \) is measurable. With \( \Phi_s \) having measurable inverse, for any \( A \in \mathcal{S} \) and \( s \geq 0 \) there exists \( \tilde{A}_s \in \tilde{S} \) such that \( A = \Phi_s^{-1}(\tilde{A}_s) \), so
the sets of the form \( \{X_u(t_1) \in A_1, \ldots, X_u(t_n) \in A_n\} \) with \( A_i = \Phi_{t_i}^{-1}A_i \) and \( t_i \leq t \) generate \( \mathcal{F}_X^{(t)} \). Consequently, \( \mathcal{F}_t = \mathcal{F}_X^{(t)} \).

Next, let \( p_{st} : \mathcal{S} \times \mathcal{S} \to [0,1] \) denote the transition probabilities of the Markov process \( (X_t, \mathcal{F}_t^X, t \geq 0) \). Then, for any \( s < t \),

\[
P(Y_t \in B | \mathcal{F}_s^X) = P(X_u(t) \in \Phi_{t}^{-1} B | \mathcal{F}_u^X) = p_{st}(Y_s, B)
\]

where

\[
p_{st} : \mathcal{T} \times \mathcal{T} \to [0,1], \quad (y, B) \mapsto p_{u(s), u(t)}(\Phi_{s}^{-1} y, \Phi_{t}^{-1} B)
\]

It is easy to see that \( p_{st} \) is a valid transition probability. Indeed, for each \( y \in \mathcal{T} \) fixed, \( p_{st}(y, \cdot) \) is a probability measure on \( \mathcal{T} \) since it is the push-forward via \( \Phi_t \) of \( p_{u(s), u(t)}(\Phi_{s}^{-1} y, \cdot) \). Similarly, for each \( B \in \mathcal{T} \), \( p_{st}(\cdot, B) \) is \( \mathcal{T} \)-measurable since it is the composition of the measurable mappings \( \Phi_{s}^{-1} : \mathcal{T} \to \mathcal{T} \) and \( p_{u(s), u(t)}(\cdot, \Phi_{t}^{-1} B) : \mathcal{T} \to [0,1] \). Finally, we verify the Chapman-Kolmogorov equations:

\[
p_{t_1t_2t_3}(y, B) = \int_\mathcal{T} p_{t_2t_3}(y, B)p_{t_1t_2}(y, dy') \]

\[
= \int_\mathcal{T} p_{u(t_2)u(t_3)}(x, \Phi_{t_3}^{-1} B)p_{u(t_1)u(t_2)}(\Phi_{t_2}^{-1} y, dx)
\]

\[
= p_{u(t_1)u(t_2)}p_{u(t_2)u(t_3)}(\Phi_{t_3}^{-1} y, \Phi_{t_2}^{-1} B)
\]

\[
= p_{u(t_1)u(t_3)}(\Phi_{t_3}^{-1} y, \Phi_{t_2}^{-1} B) = p_{t_1t_3}(y, B).
\]

This proves that \( (Y_t, \mathcal{F}_t^X)_{t \geq 0} \) is a Markov process on state space \( (\mathcal{T}, \mathcal{T}) \) with transition probabilities \( p_{st} \).

(b) By part (a) with \( \Phi_t \equiv \Phi_0 \), \( u(t) \equiv t \) and \( p_{st}(\cdot, \cdot) = p_{t-s}(\cdot, \cdot) \) we have that \( (Z_t, \mathcal{F}_t^Z)_{t \geq 0} \) is a Markov process of transition probabilities

\[
q_{st}(z, B) = p_{st}(\Phi_0^{-1} z, \Phi_0^{-1} B) = p_{t-s}(\Phi_0^{-1} z, \Phi_0^{-1} B),
\]

hence \( q_{st}(\cdot, \cdot) = q_{t-s}(\cdot, \cdot) \) and \( (Z_t, \mathcal{F}_t^Z) \) is also a homogeneous Markov process.

Additional solutions provided.

**Solution.** 8.2.32

(a) Recall part (b) of Exercise 8.1.10 that \( \theta_u := u + \tau \) are bounded \( \mathcal{F}_t \)-stopping times. Hence,

\[
\mathcal{G}_u = \mathcal{F}_{\theta_u} = \{A \in \mathcal{F}_\infty : A \cap \{u + \tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}
\]

\[
= \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq s\} \in \mathcal{F}_{s+u}, \forall s \geq 0\}.
\]

Clearly, the \( \sigma \)-algebras \( \mathcal{G}_u \) are non-decreasing in \( u \), hence \( \{\mathcal{G}_t, t \geq 0\} \) is a filtration. Further, by definition

\[
\mathcal{G}_{u+} = \bigcup_{\varepsilon > 0} \mathcal{G}_{u+\varepsilon} = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq s\} \in \mathcal{F}_{s+u+\varepsilon}, \forall s \geq 0, \varepsilon > 0\}
\]

\[
= \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq s\} \in \mathcal{F}_{(s+u)^+}, \forall s \geq 0\}.
\]

Comparing these two identities note that our assumption that \( \mathcal{F}_{s+u} = \mathcal{F}_{(s+u)^+} \) for all \( s, u \geq 0 \), yields that \( \mathcal{G}_u = \mathcal{G}_{u+} \) for all \( u \geq 0 \), as claimed.

(b) Since \( \tau \leq c \) for some non-random \( c < \infty \) we know from Proposition 8.1.13 that \( X_{\theta_t} \in m \mathcal{F}_t = m \mathcal{G}_t \) for all \( t \geq 0 \), so \( \{Y_t = X_{\theta_t} - X_{\theta_{t-}}, t \geq 0\} \) is \( \mathcal{G}_t \)-adapted. Further, \( Z_s = |X_s| \) is a right-continuous \( \mathcal{F}_s \)-sub-MG (see Exercise 8.2.9), and \( X_{u\land\theta_t} = X_{\theta_t} \) for the non-random \( u = c + t \) (since \( \theta_t = \tau + t \leq u \)). Hence, by Doob's optional
stopping theorem \( \mathbb{E}[Z_u | \mathcal{F}_{\theta_u}] \geq Z_{\theta_u} \) (see part (a) of Exercise \[8.2.30\]), and taking the expectation of both sides we deduce that \( \mathbb{E}[X_{\theta_u}] \leq \mathbb{E}[X_u] \) is finite for any \( t \geq 0 \). That is, \( \{Y_t\} \) is also an integrable S.P. of right-continuous sample functions.

Fixing \( t \geq s \geq 0 \), upon applying the same reasoning for the right-continuous sub-MG \((X_v, \mathcal{F}_v)\) and the \( \mathcal{F}_v \)-stopping times \( \theta_s \leq \theta_t \) (which are both bounded by \( u \)), we find that \( \mathbb{E}[X_{\theta_t} | \mathcal{F}_{\theta_s}] \geq X_{\theta_s} \). Consequently, for any \( t \geq s \geq 0 \),

\[
\mathbb{E}[Y_t | \mathcal{G}_s] = \mathbb{E}[X_{\theta_t} - X_{\theta_s} | \mathcal{F}_{\theta_s}] = \mathbb{E}[X_{\theta_t} | \mathcal{F}_{\theta_s}] - X_{\theta_s} \geq X_{\theta_s} - X_{\theta_s} = Y_s ,
\]

hence \( \{Y_t, \mathcal{G}_t\} \) is a right-continuous sub-MG, as claimed.

**Solution.** \[8.2.38\]

(a). With \( W_i(t), i = 1, \ldots, k \), independent standard Brownian motions,

\[
M_t = R_t^2 - kt = \sum_{i=1}^{k} (W_i(t))^2 - t
\]
is clearly integrable, of continuous sample functions and adapted to the filtration \( \mathcal{F}_t^W = \sigma(\mathcal{F}_t^W, i = 1, \ldots, k) \). By Proposition \[4.2.3\] and the mutual independence of \( \mathcal{F}_t^W, i = 1, \ldots, k \), each of the martingales \((W_i(t))^2 - t, t \geq 0)\) is also an \( \mathcal{F}_t^W \)-martingale. Consequently, so is their sum \( M_t \), as for any \( 0 \leq s \leq t \) by linearity of the C.E.

\[
\mathbb{E}[M_t | \mathcal{F}_s^W] = \sum_{i=1}^{k} \mathbb{E}[W_i(t)^2 - t | \mathcal{F}_s^W] = \sum_{i=1}^{k} (W_i(s)^2 - s) = M_s .
\]

Next, from Proposition \[8.1.15\] we know that \( \theta_{b}, \) being the first hitting time of the closed set \( B = [b, \infty) \) by the \( \mathcal{F}_t^W \)-adapted process \( R_t \) of continuous sample functions, is an \( \mathcal{F}_t^W \)-stopping time. Further, since \( R_t \geq |W_1(t)| \), clearly \( \theta_b \leq \tau_{b, b}(0) \) is a.s. finite (in view of part (a) of Exercise \[8.2.30\]).

(b). Applying part (a) of Exercise \[8.2.30\] to the continuous \( \mathcal{F}_t^W \)-MG \( M_t \) and stopping times \( \theta_b \geq 0 \), we deduce that \( \mathbb{E}[M_{u \wedge \theta_b} | \mathcal{F}_0^W] = M_0 = 0 \) for any non-random \( u \geq 0 \). Since \( M_t = R_t^2 - kt \) and \( \mathcal{F}_0^W = \{\Omega, \emptyset\} \), by linearity of the expectation this amounts to

\[
\mathbb{E}[R_{u \wedge \theta_b}^2] = k \mathbb{E}[u \wedge \theta_b]
\]

holding for all \( u \geq 0 \). By continuity of \( t \mapsto R_t \) and the definition of \( \theta_b \) we know that \( u \mapsto R_{u \wedge \theta_b} \) is uniformly bounded by \( b^2 \), hence U.I. and if \( \theta_b \) is finite (which by part (a) occurs w.p.1.), then it converges to \( R_{\theta_b}^2 \) as \( u \to \infty \). Further, \( u \wedge \theta_b \uparrow \theta_b \) as \( u \uparrow \infty \), so by U.I. and monotone convergence,

\[
\mathbb{E}[R_{\theta_b}^2] = \lim_{u \to \infty} \mathbb{E}[R_{u \wedge \theta_b}^2] = k \lim_{u \to \infty} \mathbb{E}[u \wedge \theta_b] = k \mathbb{E}\theta_b ,
\]

and upon noting that \( R_{\theta_b} = b \) (by continuity of \( t \mapsto R_t \)), we are done.
Homework 5

SOLUTION. 8.3.9 Recall Definition 8.3.8 that $\nu$ is invariant iff for any $s \geq 0$ we have $P_\nu = P_\nu \circ (\theta_s)^{-1}$. By Proposition 7.1.3 on a $\mathcal{B}$-isomorphic state space $(\mathcal{S}, \mathcal{S})$, the measures $P_\nu$ and $P_\nu \circ (\theta_s)^{-1}$ on $(\mathcal{S}, \mathcal{S})$, $\mathbb{T} = [0, \infty)$, are uniquely determined by their f.d.d. Now, recall Theorem 8.3.2 that the f.d.d. of $P_\nu$ are given, for $0 < s_1 < \cdots < s_n$ and $A \in \mathcal{S}^{n+1}$, by

$$\mu_{0,s_1,\ldots,s_n}(A) = \nu \otimes p_{s_1} \otimes p_{s_2-s_1} \otimes \cdots \otimes p_{s_n-s_{n-1}}(A).$$

By definition of the time shift $\theta_s$, the corresponding f.d.d. of $P_\nu \circ (\theta_s)^{-1}$ are then

$$\mu_{s,s+s_1,\ldots,s+s_n}(A) = \nu \otimes p_s \otimes p_{s_1} \otimes p_{s_2-s_1} \otimes \cdots \otimes p_{s_n-s_{n-1}}(\mathcal{S} \times A).$$

Consequently, $\nu$ is invariant iff for any $s \geq 0$ and $0 < s_1 < \cdots < s_n$ the identity $\mu_{0,s_1,\ldots,s_n} = \mu_{s,s+s_1,\ldots,s+s_n}$ holds. That is, $\nu$ is invariant iff the homogeneous Markov process $(X_t, t \geq 0)$ of law $P_\nu$ is such that

$$(X_0, X_{s_1}, \ldots, X_{s_n}) \overset{D}{=} (X_s, X_{s+s_1}, \ldots, X_{s+s_n}),$$

which by definition is equivalent to the stationarity of this S.P.

Next note that $\nu = \nu p_s$ is equivalent to $\nu(A) = \nu \otimes p_s(\mathcal{S} \times A)$ for all $A \in \mathcal{S}$, that is to $\mu_{0,s_1,\ldots,s_n} = \mu_{s,s+s_1,\ldots,s+s_n}$ holding (only) in case $n = 0$. However, it is not hard to check that Lemma 6.1.21 (and its proof), extends to cover arbitrary transition probabilities (indeed, such extension is to replace Lemma 6.1.21 in the 2011 version of lecture notes). Therefore, the condition $\nu = \nu p_s$ implies also that

$$\nu \otimes p_{s_1} \otimes p_{s_2-s_1} \otimes \cdots \otimes p_{s_n-s_{n-1}}(A) = \nu \otimes p_s \otimes p_{s_1} \otimes p_{s_2-s_1} \otimes \cdots \otimes p_{s_n-s_{n-1}}(\mathcal{S} \times A),$$

that is, $\mu_{0,s_1,\ldots,s_n} = \mu_{s,s+s_1,\ldots,s+s_n}$. We thus conclude that the condition $\nu = \nu p_s$ for all $s \geq 0$, is equivalent to invariance of $\nu$.

SOLUTION. 8.3.10

(a). The S.P. $Y_t = \Phi_0(W_t)$ for the invertible map $\Phi_0(x) = e^x : \mathbb{R} \to (0, \infty)$ is a homogeneous Markov process (of state space $(0, \infty)$), in view of part (b) of Exercise 8.3.4.

The S.P. $Z_t^{(r,0)} = rt$ is non-random, hence a Markov process, whereas if $\sigma \neq 0$, then $Z_t^{(r,\sigma)} = \Phi_t(W_t)$ for the invertible maps $\Phi_t(x) = \sigma x + rt : \mathbb{R} \to \mathbb{R}$ is a Markov process in view of part (a) of Exercise 8.3.4.

Likewise, the S.P. $U_t = \Phi_t(W_{t+1})$ for the invertible maps $\Phi_t(x) = e^{-t/2}x$ and the strictly increasing, invertible $u(t) = e^{t^2} : [0, \infty) \to [0, \infty)$, is a Markov process (by part (a) of Exercise 8.3.4) and the fact that $\{W_{t+1}, t \geq 0\}$ is a Markov process.

The standard Brownian bridge ($\tilde{B}_t, t \in [0,1]$) is also a Markov process. Indeed, recall part (c) of Exercise 7.3.10 that this S.P. is of the form $\tilde{B}_t = \Phi_t(W_{u(t)})$ for the invertible maps $\Phi_t(x) = (1-t)x$, $t \in [0,1]$ and the strictly increasing, invertible $u(t) = t/(1-t) : [0,1) \to [0, \infty)$ (while $\tilde{B}_t = 0$ is non-random, hence does not affect the Markov property).

Proceeding to check which of the Markov processes $Z_t^{(r,\sigma)}$, $U_t$ and $\tilde{B}_t$ is also homogeneous, note that the maps $\Phi_t(\cdot)$ have been linear in all three cases, so each is a Gaussian process. The transition probabilities $\{p_{s,t}(\cdot, \cdot), t > s \geq 0\}$ of any Gaussian Markov process $(X_t, t \geq 0)$ are clearly given by $p_{s,t}(m, B)$ of (8.3.4), with $\nu = \nu_X(s, t) = \text{Var}(X_t | X_s)$ and $m = m_X(s, t, x) = \mathbb{E}[X_t | X_s]$ evaluated at $X_s = x$.

It thus remains only to check in which of the three cases under consideration both $m(s, t, x)$ and $\nu(s, t)$ depend only on $t - s$ (and possibly $x$).
To this end, from the representation

\[ Z_t^{(r, \sigma)} = Z_s^{(r, \sigma)} + r(t - s) + \sigma(W_t - W_s) \]

and the independence of \( W_t - W_s \) from \( F_s \), we easily deduce that \( m_{Z_t}(s, t, x) = x + r(t - s) \) and \( \nu_{Z_t}(s, t) = \sigma^2(t - s) \). Consequently, the Markov process \( (Z_t^{(r, \sigma)}, t \geq 0) \) is indeed homogeneous.

Similarly, we have the representation

\[ U_t = e^{-(t-s)/2}U_s + e^{-t/2}(W_{e^t} - W_e^t). \]

As the increment \( W_{e^t} - W_e^t \) is independent of \( F_{e^t} \), we find that \( m_{U_t}(s, t, x) = e^{-(t-s)/2}x \) and \( \nu_{U_t}(s, t) = 1 - e^{-(t-s)} \). Therefore, the Markov process \((U_t, t \geq 0)\) is also homogeneous.

In contrast, by the same reasoning we deduce from the representation

\[ \tilde{B}_t = \frac{1-t}{1-s} \tilde{B}_s + \left(1 - t\right)(W_t/(1-t) - W_s/(1-s)) \]

and independence between \( \tilde{B}_s \) and \( W_t/(1-t) - W_s/(1-s) \), that \( m_{\tilde{B}_t}(s, t, x) = (1 - t)x/(1-s) \) (and \( \nu_{\tilde{B}_t}(s, t) = (1-t)^2[t/(1-t) - s/(1-s)] \)). Hence, the Markov process \((\tilde{B}_t, t \in [0,1])\) is non-homogeneous.

(b). One such example is the S.P. \( \{Y_t, t \geq 0\} \) which by part (a) is a homogeneous Markov process. Indeed, for \( t > s \) we have the representation \( Y_t - Y_s = X_{(1-t)/(1-s)} - X_{s/(1-s)} \), where \( X_{s/t} = \text{Feller semi-group} \) is independent of \( Y_s \). In particular, \( E[Y_t - Y_s] = E[Y_t]E[D_{s,t}] = e^{s/2}(e^{(t-s)/2} - 1) \) depends on both \( s \) and \( t-s \), hence the S.P. \( \{Y_t, t \geq 0\} \) does not have stationary increments. Further, \( E[Y_t - Y_s|Y_s] = Y_s(e^{(t-s)/2} - 1) \) is of positive variance, in contradiction to possible independence of \( Y_t - Y_s \) and \( Y_s \), so the S.P. \( \{Y_t, t \geq 0\} \) does not have independent increments.

(c). One such example is the Gaussian process \( \{\tilde{B}_t, t \in [0,1]\} \) which by part (a) is not a homogeneous Markov process. Indeed, recall part (c) of Exercise 7.3.16 that this S.P. has zero mean and the auto-covariance function \( c(s,t) = s \wedge t - st \). Thus, fixing \( 0 \leq s < t \leq 1 \), the increment \( \tilde{B}_t - \tilde{B}_s \) is by linearity of the expectation, a Gaussian variable of zero mean, whose variance

\[ E[(\tilde{B}_t - \tilde{B}_s)^2] = (t-s) + c(s,t) - 2c(s,t) = t - t^2 + s - t^2 - 2s(1-t) = (t-s) - (t-s)^2, \]

depends only on \( t-s \). Consequently, this S.P. has stationary increments.

**Solution.** 8.3.20

(a). Recall Proposition 8.3.19 that if real valued S.P. \( \{X_t, t \geq 0\} \) has stationary, independent increments, then it is a homogeneous Markov process, whose Markov semi-group is \( p_t(y, \cdot) = P(y + Z(\cdot)) \), for the random variable \( Z = X_t - X_0 \). Consequently, in this case \( (p_t f)(x) = E[f(x + Z)] \) for any bounded Borel function \( f \) and all \( x \in \mathbb{R} \). If \( f \in C_b(\mathbb{R}) \) and \( x_n \to x \) then \( f(x_n + Z(\omega)) \overset{a.s.}{\to} f(x + Z(\omega)) \), so by bounded convergence \( (p_t f)(x_n) = E[f(x_n + Z)] \to E[f(x + Z)] = (p_t f)(x) \). Furthermore, \( \sup_x |(p_t f)(x)| \leq \sup_y |f(y)| \) is finite, so as claimed \( p_t \) maps \( C_b(\mathbb{R}) \) to \( C_b(\mathbb{R}) \) (see Definition 8.3.18 of a Feller semi-group).

(b). By Proposition 8.3.19 the additional property of having right-continuous sample functions is sufficient for a Markov process \( (X_t, F_t) \) with a Feller semi-group, to be a strong Markov process. Since the Poisson process and any Brownian Markov process have stationary, independent increments and right-continuous sample functions, these are thus strong Markov processes.
(c). Fixing $t$ finite, let $m = \lceil 2^\ell \rceil + 1$, setting $t_m = t$ and $t_k = k2^{-\ell}$ for $k = 0, 1, \ldots, m - 1$. Then, since $X_0 = 0$, 
$$
|X_t| = |X_t - X_0| \leq \sum_{k=1}^m |X_{t_k} - X_{t_{k-1}}|.
$$

By our assumption that $\mathbb{E}[X_t] \to 0$, for some integer $\ell$ large enough, we have that $\mathbb{E}[X_s]$ is finite for all $s \in [0, 2^{-\ell}]$. Further, having stationary increments, $X_{t_k} - X_{t_{k-1}} \overset{D}{=} X_{s_k}$ where $s_k = t_k - t_{k-1}$ is in $[0, 2^{-\ell}]$, hence $\mathbb{E}[X_t] \leq \sum_{k=1}^m \mathbb{E}[X_{s_k}]$ is finite for all $t$ finite.

In particular, recall Proposition 8.2.4 that $M_t = X_t - \mathbb{E}X_t$, being an integrable S.P. of independent increments and constant (zero) mean is a martingale. Setting $g(t) = \mathbb{E}X_t$, we further claim that $g(t) = tg(1)$, namely, that $M_t = X_t - t\mathbb{E}X_1$.

Indeed, by stationarity of the increments $X_{u+s} - X_u$ we have that $g(u + s) = g(u) + g(s)$ for all $u, s \geq 0$ and consequently, $g(q) = qg(1)$ for any $q \in \mathbb{Q}^2$.

To extend this claim to $t$, consider $q_k \in \mathbb{Q}^2$ such that $q_k \downarrow t$, noting that by stationarity of the increments $X_{q_k} - X_t$ we have that 
$$
|g(t) - g(q_k)| \leq \mathbb{E}[X_{q_k} - X_t] = \mathbb{E}|X_{q_k} - X_t|
$$
which converges to zero as $k \to \infty$, by our assumption that $\mathbb{E}[X_u] \to 0$ when $u \downarrow 0$.

Fixing an integrable $\mathcal{F}_t$-stopping time $\tau$, we proceed to show that $M_\tau$ is an integrable random variable and $\mathbb{E}[M_\tau] = 0$, from which we directly get that $X_\tau$ is an integrable random variable and $\mathbb{E}X_\tau = \mathbb{E}[\mathbb{E}X_1]$. To this end, fixing $r \geq 1$ let $s_k = k2^{-\ell}$ for $k \in \mathbb{Z}_+$ and consider the integrable $\mathcal{F}_{s_k}$-stopping times $\tau_{s_k} = 2^{-\ell}([2^\ell \tau] + 1)$, $\ell = 0, 1, \ldots, r$ of Lemma 8.1.10 (which are non-increasing in $\ell$). The discrete-time martingale $\{M_{s_k}, s_k, k \in \mathbb{Z}_+\}$ has i.i.d. integrable differences, hence the stopped MG $\{M_{s_k \land \tau_0}, k \in \mathbb{Z}_+\}$ is uniformly integrable (see part (a) of Proposition 5.4.3), and so by Doob’s optional stopping, for any $\mathcal{F}_{s_k}$-stopping time $\theta \leq \tau_0$ the R.V. $M_\theta$ is integrable and $\mathbb{E}M_\theta = \mathbb{E}M_0 = 0$ (see Theorem 5.4.1). Just as in the proof of Theorem 8.2.20 this implies that $(M_{\tau_{s_k}} \mathcal{F}_{s_k}, n \in \mathbb{Z}_+)$ forms a discrete-time RMG, hence by Lévy’s downward theorem the sequence $M_{\tau_{s_k}} = \mathbb{E}[M_{s_k} | \mathcal{F}_{s_k}]$ is U.I. By its definition, $\tau_{s_k} \downarrow \tau$, which in view of the right-continuity of $t \mapsto M_t(\omega)$ yields the convergence $M_{\tau_{s_k}} \overset{a.s.}{\to} M_{\tau}$ as $\ell \to \infty$. With $\{M_{\tau_{s_k}}\}$ U.I. and of zero mean, this in turn implies that $M_\tau$ is also an integrable, zero-mean, R.V. as claimed.

**Solution.** 8.3.22 (a) From Remark 8.3.3 and Definition 8.3.7 we have that conditional on $X_0 = x$, the law of a Brownian Markov process $(X_t, \mathcal{F}_t)$ is the same as that of the Brownian motion. Hence, the $\mathcal{F}_t$-adapted process $\{M_t\}$ is integrable. Further, for $0 \leq s < t$, we have 
$$
\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[M_t e^{i\theta(X_t - X_s) + (t-s)\theta^2/2} | \mathcal{F}_s] = M_t \mathbb{E}[e^{i\theta(X_t - X_s) + (t-s)\theta^2/2}] = M_s,
$$
where we have used the independence of $X_t - X_s$ and $\mathcal{F}_s$ and the known characteristic function of the normal r.v. $X_t - X_s$. Taking real and imaginary parts, we obtain the martingale property for $(R_t, \mathcal{F}_t)$ and $(I_t, \mathcal{F}_t)$.

(b) Recall Definition 8.3.7 that the sample functions $t \mapsto X_t$ are continuous and hence so are $t \mapsto M_t$. Since the $\mathcal{F}_t$-Markov time $\tau$ is bounded, $\tau + u \leq T$ for some non-random, finite $T$. Hence, applying Doob’s optional stopping theorem (see Corollary 8.2.27), for the MG $(M_{\tau+u}, \mathcal{F}_{\tau+u})$ of last element and the Markov times $\tau + u \geq \tau$, we get that $\mathbb{E}[M_{\tau+u} | \mathcal{F}_{\tau+u}] = M_{\tau}$. 

310C: HOMEWORK SOLUTIONS 2017 411
(c) On event $\tilde{\Omega} \in \mathcal{F}_{\tau+}$ of probability one, and for all $\theta \in \mathbb{Q}$,

$$e^{i\theta X_{\tau+} - u\theta^2/2} = \mathbb{E}[e^{i\theta X_{\tau+}} | \mathcal{F}_{\tau+}] = \int e^{i\theta x} d\tilde{P}_{X_{\tau+}|\mathcal{F}_{\tau+}}(x, \omega),$$

where the first identity comes from part (b) and the second comes from Exercise 4.4.6. By continuity in $\theta$ of both sides (see Proposition 3.3.2(d)), same identities hold for all $\omega \in \tilde{\Omega}$ and $\theta \in \mathbb{R}$. Recall Example 3.3.6 that the left side is the characteristic function of the normal distribution of mean $X_\tau(\omega)$ and variance $u$. Accordingly, by Lévy’s inversion theorem (see Theorem 3.3.14), the R.C.P.D. $\tilde{P}_{X_{\tau+}|\mathcal{F}_{\tau+}}$ coincides a.s. with this normal distribution.

(d) By the definition of R.C.P.D. and part (c), for the Brownian semi-group $\{p_u, u \geq 0\}$ of (8.3.10) and any Borel set $B$, a.s.

$$\mathbb{P}[X_{\tau+} \in B | \mathcal{F}_{\tau+}] = \tilde{P}_{X_{\tau+}|\mathcal{F}_{\tau+}}(B) = p_u(X_\tau, B).$$

With this holding for any $u \geq 0$ and bounded $\mathcal{F}_\tau$-Markov time $\tau$, we verified the conditions of Proposition 8.3.15 from which it follows that the Brownian Markov process has the strong Markov property.

**Solution.**

(a)-(f). Since $W_t$ is Gaussian, each of its finite dimensional distributions is multivariate normal. The multivariate normal law is preserved under linear combinations of the R.V.-s, hence each of the S.P. in (a)-(c) is a Gaussian process. Similarly, (f) is a Gaussian process since its f.d.d.-s correspond to sums of independent random vectors, each of which is multivariate normal, thus leading to multivariate normal f.d.d.-s in case (f).

The continuity of the sample functions is obviously preserved in cases (a)-(d) and (f), whereas recall part (b) of Exercise 7.3.10 that a.s. $\lim_{s \to 0} W_t^{(5)} = \lim_{s \to \infty} s^{-1}W_s = 0$, hence $\tilde{W}_t^{(5)}$ is indistinguishable from a S.P. of continuous sample functions.

Clearly, $\mathbb{E}[\tilde{W}_t^{(i)}] = 0$ for $i = 1, \ldots, 6$ and all $t \geq 0$. So with $c(s,t) = s \land t$ denoting the auto-covariance of the standard Brownian motion, it remains only to verify that $c_i(s,t) = \text{Cov}(\tilde{W}_t^{(i)}, \tilde{W}_t^{(j)}) = c(s,t)$ for $i = 1, \ldots, 6$ and $s \geq 0$. We proceed with this task, utilizing in each case the bi-linearity of Cov($\cdot$,$\cdot$):

1. $c_1(s,t) = \text{Cov}(-W_s, -W_t) = \text{Cov}(W_s, W_t) = c(s,t)$.
2. $c_2(s,t) = c(T + s, T + t) - c(T + s, T) - c(T + t, T) + c(T, T)$
   $= T + (s \land t) - T - T + T = s \land t$.
3. Taking $0 \leq s, t \leq T$, here $c_3(s,t) = (T, T) - c(T, T - t) - c(T - s, T) + c(T - s, T - t)$
   $= T - (T - t) - (T - s) + T - s \lor t = s + t - s \lor t = s \land t$.
4. $c_4(s,t) = \text{Cov}(\alpha^{-1/2}W_{as}, \alpha^{-1/2}W_{at}) = \alpha^{-1}\text{Cov}(W_{as}, W_{at})$
   $= \alpha^{-1}(a \land at) = s \land t$.
5. Clearly $c_5(0,t) = c_5(t,0) = 0$ since $\tilde{W}_0^{(5)} = 0$, while for $s,t$ positive, $c_5(s,t) = \text{Cov}(sW_{1/s}, tW_{1/t}) = st \text{Cov}(W_{1/s}, W_{1/t}) = st(s \land t) = t \land s$.
(6). Since $W_t^{(k)}$ are independent and of zero mean, $\text{Cov} \left( W_t^{(k)}, W_t^{(\ell)} \right) = 0$ for $k \neq \ell$, hence
\[
c_0(s, t) = \text{Cov} \left( \sum_{k=1}^{n} c_k W_s^{(k)}, \sum_{\ell=1}^{n} c_{\ell} W_t^{(\ell)} \right) = \sum_{k, \ell=1}^{n} c_k c_{\ell} \text{Cov} \left( W_s^{(k)}, W_t^{(\ell)} \right)
\]
\[
= \sum_{k=1}^{n} c_k^2 \text{Cov} \left( W_s^{(k)}, W_t^{(k)} \right) = \sum_{k=1}^{n} c_k^2 (s \land t) = s \land t.
\]

(g). Since $\tilde{W}_t^{(3)}$, $t \in [0, T]$ are measurable on $\mathcal{F}_T^W$, clearly $\mathcal{F}_T^{\tilde{W}^{(3)}} \subseteq \mathcal{F}_T^W$. Each of the Brownian increments $\tilde{W}_t^{(2)} = W_{T+t} - W_T$ is independent of $\mathcal{F}_T^W$, hence by definition so is $\mathcal{F}_T^{\tilde{W}^{(2)}}$. That is, the S.P. $\{\tilde{W}_t^{(2)}, t \geq 0\}$ is independent of the S.P. $\{\tilde{W}_t^{(3)}, t \in [0, T]\}$.

Next, note that $\{W_T > W_{T-t} > W_{T+t}\} = \{0 > \tilde{W}_t^{(3)} > \tilde{W}_t^{(2)}\}$. Writing $N = \tilde{W}_t^{(2)}$ and $M = -\tilde{W}_t^{(3)}$ which are i.i.d. $\mathcal{N}(0, t)$ random variables, in case $t > 0$,
\[
q_t = \mathbf{P}[0 > M > N] = \mathbf{P}[M > N|M < 0, N < 0]\mathbf{P}[M < 0]\mathbf{P}[N < 0] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8},
\]
while by definition $q_0 = 0$.

Additional solutions provided.

**Solution.** 3.3.17

(a). Though the set $\mathbb{C}$ of possible values of $\tau$ is countable, it may have accumulation points (for example, consider $\mathbb{C} = \mathbb{Q}^{(2)}$), so you cannot assume that $s_1 < s_2 < \cdots < s_k < \cdots$. Nevertheless, this part of the exercise holds for any $\mathcal{F}_\tau$-stopping time. Indeed, recall part (a) of Exercise 3.1.11 that $\mathcal{F}_\tau$ is a $\sigma$-algebra and $\sigma(\tau) \subseteq \mathcal{F}_\tau$. Hence, if $A \in \mathcal{F}_\tau$ and $t \geq 0$ is finite, non-random, then also $B = A \cap \{\tau = t\} \in \mathcal{F}_\tau$.

With $\tau$ an $\mathcal{F}_\tau$-stopping time, this implies that $B = B \cap \{\tau \leq t\} \in \mathcal{F}_\tau$.

By the same reasoning, $Y I_{\{\tau = t\}} \in b\mathcal{F}_\tau$ provided $Y \in b\mathcal{F}_\tau$ and $t \geq 0$ is non-random.

(b). Since $\mathbb{C}$ is countable, with $\Sigma_\ast$ denoting the sum over finite $s_k \in \mathbb{C}$, we have the decomposition
\[
\begin{equation}
\begin{aligned}
\quad h(\tau, X_{\tau+}) I_{\{\tau < \infty\}} = \Sigma_\ast \left( h(s_k, X_{s_k+}) I_{\{\tau = s_k\}} \right).
\end{aligned}
\end{equation}
\]

Being a Markov process on $\mathcal{B}$-isomorphic state space $(\mathbb{S}, \mathcal{S})$, we know that $X_{s+} : (\Omega, \mathcal{F}) \mapsto (\mathbb{S}\{0, \infty\}, \mathcal{S}\{0, \infty\})$ is measurable for any non-random $s \geq 0$. By assumption $h(s_k, \cdot)$ is $\mathcal{S}\{0, \infty\}$-measurable, so due to the closure of R.V.-s to compositions we deduce that $h(s_k, X_{s_k+})$ is a R.V. for any finite $s_k \in \mathbb{C}$. Clearly, $\{\tau = s_k\}$, $s_k \in \mathbb{C}$, are disjoint events in $\mathcal{F}$, so by (*) we conclude that $h(\tau, X_{\tau+}) I_{\{\tau < \infty\}}$ is also a R.V. on $\Omega, \mathcal{F}$.

Next, the assumed uniform boundedness of $h(\cdot, \cdot)$ on $\mathbb{C} \times \mathbb{S}\{0, \infty\}$ results with $g_h(s_k, \cdot) = \mathbb{E}_\tau h(s_k, \cdot)$ which is uniformly bounded on $\mathbb{C} \times \mathbb{S}$. Further, $g_h(s_k, X_{s_k}) \in b\mathcal{F}_{s_k}$, so $g_h(s_k, X_{s_k}) I_{\{\tau = s_k\}} \in b\mathcal{F}_\tau$ (as mentioned in solution of part (a)), and from the decomposition
\[
g_h(\tau, X_{\tau}) I_{\{\tau < \infty\}} = \Sigma_\ast g_h(s_k, X_{s_k}) I_{\{\tau = s_k\}}, \tag{**}
\]
we conclude that $g_h(\tau, X_{\tau}) I_{\{\tau < \infty\}} \in b\mathcal{F}_\tau$, as claimed.

(c). Fix $A \in \mathcal{F}_\tau$ and finite $s_k \in \mathbb{C}$. Then, by the tower property, upon taking out the known $I_{A\cap\{\tau = s_k\}}$ (per part (a)) and applying the Markov property at time $s_k$,
we find that
\[
\mathbb{E}[h(s_k, X_{s_k+})I_{A \cap \{\tau = s_k\}}] = \mathbb{E}[\mathbb{E}[h(s_k, X_{s_k+})|F_{s_k}]I_{A \cap \{\tau = s_k\}}] = \mathbb{E}[g_h(s_k, X_{s_k})I_{A \cap \{\tau = s_k\}}].
\]
Summing this identity over all finite \( s_k \in \mathbb{C} \) we deduce in view of (*) and (**) that
\[
\mathbb{E}[h(\tau, X_{\tau+})I_{\{\tau < \infty\}}I_A] = \sum \mathbb{E}[g_h(s_k, X_{s_k})I_{A \cap \{\tau = s_k\}}] = \mathbb{E}[g_h(\tau, X_\tau)I_{\{\tau < \infty\}}I_A].
\]
This applies to all \( A \in \mathcal{F}_\tau \), so with \( g_h(\tau, X_\tau)I_{\{\tau < \infty\}} \in b\mathcal{F}_\tau \) the a.s. validity of (8.3.10) follows by the definition of C.E.

**Solution. 9.1.7**

(a). For \( t = 1 \) and \( \tau = 2I_{\{W_2 > 0\}} \) we have that,
\[
\mathbb{P}(W_{\tau+t} - W_\tau > 0) = \mathbb{P}(W_2 > 0, W_3 - W_2 > 0) + \mathbb{P}(W_2 \leq 0, W_1 > 0).
\]
Since the Brownian motion has stationary independent increments \( \mathbb{P}(W_2 > 0, W_3 - W_2 > 0) = \mathbb{P}(W_2 > 0)\mathbb{P}(W_1 > 0) = 1/4 \). Further, \( W_2 = W_1 + (W_2 - W_1) \) with \( W_2 - W_1 \) a standard normal variable, independent of \( W_1 \). Hence, it is easy to verify by symmetry considerations that \( \mathbb{P}(W_2 \leq 0, W_1 > 0) = 1/8 \). Consequently, \( \mathbb{P}(W_{\tau+t} - W_\tau > 0) = 3/8 \neq 1/2 = \mathbb{P}(W_t > 0) \). That is, \( \{W_{t+\tau} - W_\tau, t \geq 0\} \) does not have the law of standard Brownian motion.

(b). Consider the stopping time \( \tau = \inf\{t \geq 0 : W_t > 1\} \). Then, by sample path continuity
\[
\{[\tau] \leq 0\} = \{\tau < 1\} = \{\sup_{t \in [0,1]} W_t > 1\},
\]
which is certainly not in \( \mathcal{F}_0^W \).
Solution. \([\text{9.1.8}]\)
(a) Recall that \(\tau_b = \inf\{t \geq 0 : W_t \geq b\}\) is a.s. finite and for \(b \geq 0\), by sample path continuity of the Brownian Markov process we have that \(P_0\)-a.s. \(W_{\tau_b} = b\). Considering the S.P. \(\hat{W} = W_{\tau_b} - W_{\tau_b}\) we thus get that \(P_0\)-a.s. \(\tau_{b+} = \tau_b + \tau_{b+}(\hat{W})\). With \(\tau_b\) an \(\mathcal{F}_t\)-stopping time we have from Corollary 9.1.6 that \(\hat{W}\) is a standard Wiener process and consequently from Corollary 9.1.6 we deduce that a.s. \(\tau_{b+}(\hat{W}) = 0\). Combining these fact we conclude that \(P_0(\tau_{b+} \neq \tau_b) = 0\), as claimed.
(b) Considering the S.P. \(\tau\) identity due to our assumption that \((W, F)\) is a Brownian Markov process (and in particular \(F^W \subseteq F)\). Now, with finite valued \(H \geq 0 = W_0\) and \(\tau_H\) a finite \(\mathcal{G}_t\)-stopping time, arguing as in part (a) we find that a.s. \(W_{\tau_H} = H\) and \(\hat{W} = W_{\tau_H} - W_{\tau_H}\) being a standard Wiener process, from which we deduce that a.s. \(\tau_{H+} = \tau_H + \tau_{0+}(\hat{W}) = \tau_H\).

Solution. \([\text{9.1.11}]\)
(a) Suppose \(y > x\). Then, recall \([\text{9.1.2}]\), that
\[
\begin{align*}
P_x(T_y > u) &= P_x(M_u < y) = P_x(\sup_{0 \leq t \leq u} (W_t - W_0) < y - x) \\
&= P(\sup_{0 \leq t \leq u} \hat{W}_t < y - x) = P(T_{y-x} > u),
\end{align*}
\]

since, by definition, \(\hat{W}_t = W_t - W_0\) is a standard Wiener process. If \(y < x\) then by the symmetry of the standard Wiener process (see part (a) of Exercise 9.1.1) and the preceding \(P_x(T_y > u) = P_y(T_x > u) = P(T_{x-y} > u)\). Finally, since \(P_x\)-a.s. \(T_x = 0\), for \(x = y\) we have that \(P_x(T_y > u) = I_{\{u < 0\}} = P(T_0 > u)\).

Regarding \(R_t = \inf\{s > t : W_s = 0\}\), upon applying the Markov property of Brownian motion at time \(t\), we have by the preceding identities for \(T_y\), that for any \(t, u > 0\),
\[
P(R_t > t + u) = E[P_{W_t}(T_0 > u)] = \int_{-\infty}^{\infty} p_t(0, y)P_y(T_0 > u)dy
\]

Turning to \(L_t = \sup\{s \in [0, t] : W_s = 0\}\), we thus have that for any \(0 < u < t\),
\[
P(L_t \leq u) = P(W_u \neq 0, \forall s \in (u, t]) = P(R_u > t)
\]

(b) Since \(T_{|y|}\) has a density we have from part (a) and Fubini’s theorem that
\[
P(R_t - t > u) = \int_{-\infty}^{\infty} p_t(0, y)dy \int_{u}^{\infty} f_{T_{|y|}}(s)ds = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} p_t(0, y)f_{T_{|y|}}(s)dy.
\]
Consequently, $R_t - t$ has the density

$$f_{R_t - t}(u) = \int_{-\infty}^{\infty} p_t(0, y) f_{T_{|y|}}(u) dy.$$ 

Plugging in the values of $p_t(0, y)$ and $f_{T_{|y|}}(u)$ of (8.3.9) and (9.1.4), respectively, then performing the (hinted) change of variable (so $|z| dz = |y| dy(t + u)/(ut)$), we find that

$$f_{R_t - t}(u) = \int_{-\infty}^{\infty} \frac{|y|}{2\pi \sqrt{tu^3}} e^{-\frac{y^2}{2tu^3}} dy$$

$$= \frac{\sqrt{t}}{\sqrt{2\pi u(t + u)}} \int_{-\infty}^{\infty} \frac{|z|}{\sqrt{2\pi}} e^{-\frac{z^2}{2u}} dz = \frac{\sqrt{t}}{\pi \sqrt{u(t + u)}}$$

(recall that $\mathbf{E}[G] = \sqrt{2/\pi}$ when $G$ has the standard normal distribution).

(c). Fixing $u \in [0, t]$ and $\Theta \in [0, \pi/2]$ such that $\tan^2(\Theta) = (t - u)/u$, we deduce from part (b) that

$$F_{L_t}(u) := \mathbf{P}(L_t \leq u) = \mathbf{P}(R_u > t) = \int_{t-u}^{\infty} f_{R_u-u}(s) ds$$

$$= \frac{\sqrt{u}}{\pi} \int_{t-u}^{\infty} \frac{ds}{\sqrt{s(u + s)}} = \frac{2}{\pi} \int_{\Theta}^{\pi/2} d\theta ,$$

where the last equality follows by the change of variable $s = u \tan^2(\theta)$. Since $\sin(\pi/2 - \Theta) = \cos(\Theta) = \sqrt{u/t}$, we find as stated that

$$F_{L_t}(u) = \frac{2}{\pi} \left[ \frac{\pi}{2} - \Theta \right] = \frac{2}{\pi} \arcsin(\sqrt{u/t}).$$

Now, differentiating the identity $\sin(\pi/2 F_{L_t}(u)) = \sqrt{u/t}$ with respect to $u$ leads to the density formula,

$$f_{L_t}(u) = \frac{d}{du} F_{L_t}(u) = \frac{1}{\pi \sqrt{u(t - u)}} .$$

(d). Note that for any $u \leq t \leq v$,

$$\mathbf{P}(R_t > v, L_t \leq u) = \mathbf{P}(W_s \neq 0, \forall s \in (u, v]) = \mathbf{P}(L_v \leq u) .$$

Hence, the joint density of $(R_t, L_t)$ is

$$f_{R_t, L_t}(v, u) = -\frac{\partial^2}{\partial v \partial u} \mathbf{P}(R_t > v, L_t \leq u) = -\frac{\partial}{\partial v} f_{L_t}(u)$$

and from part (c) we conclude that

$$f_{R_t, L_t}(v, u) = \frac{1}{2\pi \sqrt{u(v - u)^3}} I_{\{0 \leq u \leq t \leq v\}} .$$

**Solution. 9.1.14 (a)** We have $\tau \equiv \tau^{(1)} \wedge \tau^{(2)}$ where $\tau^{(i)} \equiv \inf\{t \geq 0 : W_i(t) = 0\}$. Therefore, with $x_1, x_2 > 0$, $\tau^{(1)} \mathcal{D} T_{x_1}$ and $\tau^{(2)} \mathcal{D} T_{x_2}$, by the independence of $W_1(\cdot)$ and $W_2(\cdot)$,

$$\mathbf{P}(\tau > t) = \mathbf{P}(\tau^{(1)} > t) \mathbf{P}(\tau^{(2)} > t) = (2F_G(\frac{x_1}{\sqrt{t}}) - 1)(2F_G(\frac{x_2}{\sqrt{t}}) - 1),$$

and differentiating in $t$, we see that $\tau$ has density

$$1_{t \geq 0} t^{-3/2} \left[ x_2 f_G(\frac{x_2}{\sqrt{t}})(2F_G(\frac{x_1}{\sqrt{t}}) - 1) + x_1 f_G(\frac{x_1}{\sqrt{t}})(2F_G(\frac{x_2}{\sqrt{t}}) - 1) \right] .$$
(b) Setting \( a_t = -\infty \) in the identity (9.1.6) it follows that for a standard Brownian motion, \( u > 0 \) and any \( b \geq \max(0, a) \),

\[
P(T_b > u, W_u < a) = P(W_u < a) - P(W_u > 2b - a).
\]

If \( \tau = \tau^{(1)} < \tau^{(2)} \) then \( W_2(\tau) = 0 = W_1(\tau) \). Recall that \( x_i - W_i(u) \overset{D}{=} W_u \), hence by the strong Markov property of \( W_1 = (W_1, W_2) \) at \( \tau^{(1)} \) and the mutual independence of \( W_1(\cdot) \) and \( W_2(\cdot) \) we have that for any \( y \geq 0 \),

\[
P(\tau \leq t, W_1(\tau) = 0, W_2(\tau) > y) = P(\tau^{(1)} \leq t, \tau^{(2)} > \tau^{(1)}, W_2(\tau^{(1)}) > y)
\]

\[
= \int_0^t P(\tau^{(2)} > u, W_2(s) > u) f_{\tau^{(1)}}(u) \, du
\]

\[
= \int_0^t \left[ F_G\left(\frac{x_2 - y}{\sqrt{u}}\right) - (1 - F_G\left(\frac{x_2 + y}{\sqrt{u}}\right))\right] \frac{x_1}{u^{3/2}} f_G(\frac{x_1}{\sqrt{u}}) \, du,
\]

The density of \( (\tau, W_1(\tau), W_2(\tau)) \) with respect to Lebesgue measure on \( \{ (t, x, y) \in (0, \infty)^3 : x = 0 \} \) is therefore given by

\[
\left[ f_G\left(\frac{x_2 - y}{\sqrt{u}}\right) - f_G\left(\frac{x_2 + y}{\sqrt{u}}\right)\right] \frac{x_1}{t^2} f_G(\frac{x_1}{\sqrt{t}}),
\]

and merely exchanging \( x_1 \) with \( x_2 \) and \( x \) with \( y \) gives the formula for this density on the set \( \{ (t, x, y) \in (0, \infty)^3 : y = 0 \} \).

**Solution.** (9.1.10)

(a) Clearly \( \bar{M}_t \geq \bar{W}_t \), are both \( \mathcal{F}_t \)-adapted, hence \( Y_t = M_t - \bar{W}_t \) is an \( \mathbb{R}_+ \)-valued, \( \mathcal{F}_t \)-adapted S.P. Fixing \( s \geq 0 \) and \( u > 0 \), recall that \( \{ \bar{W}_t = W_{s+t} - W_s, t \geq 0 \} \) is a standard Wiener process which is independent of \( \mathcal{F}_s \) (see Exercise 9.1.11, or Corollary 9.1.6), hence the corresponding running maximum \( \bar{M}_u = \sup_{t \in [0, u]} \bar{W}_t \) is also independent of \( \mathcal{F}_s \). Clearly, \( M_{s+u} = \max(M_s, \bar{M}_u + W_s) \) and subtracting \( W_{s+u} = \bar{W}_u + W_s \) from both sides, we arrive at \( Y_{s+u} = \max(Y_s, \bar{M}_u) - \bar{W}_u \). We claim that for any \( y, c \in \mathbb{R}_+ \),

\[
(\ast) \quad P(\max(y, \bar{M}_u) - \bar{W}_u > c) = p_{+u}(y, (c, \infty)),
\]

hence by the preceding for any open interval of the form \( B = (c, \infty) \),

\[
P(Y_{s+u} \in B | \mathcal{F}_s) = p_{+u}(Y_s, B).
\]

Since both sides of this identity are (in terms of \( B \)), probability measures on \( \mathbb{R}_+ \), it extends by the \( \pi - \lambda \) theorem to all Borel subsets of \( \mathbb{R}_+ \). We are then done, for we already verified in part (b) of Exercise 9.1.11 that \( p_{+u}(\cdot, \cdot) \) is a Markov semi-group on \( \mathbb{R}_+ \).

Turning to prove (\ast), note that \( \{(a, b) : \max(y, b) - a > c\} \) is the disjoint union of the sets \( \{(a, b) : a < y - c\} \) and \( \{(a, b) : b > y, a \in [y - c, b - c]\} \). Hence, the left side of (\ast) equals to

\[
P(\bar{W}_u < y - c) + P(\bar{M}_u > y, \bar{W}_u \in [y - c, \bar{M}_u - c]),
\]

and as the first term equals \( p_u(0, (c-y, \infty)) = p_u(y, (c, \infty)) \), it suffices to show that

\[
Q := P(\bar{M}_u > y, \bar{W}_u \in [y - c, \bar{M}_u - c]) = \int_{y}^{\infty} db \int_{y-c}^{b-c} f_{\bar{W}_u, \bar{M}_u}(a, b) da
\]

\[
= p_u(y, (-\infty, -c))\]
and (b) of Exercise 9.1.18, the same law (see Theorem 8.3.2).

(a). For fixed $n$ expect that $P_{\pi T}$ so the law of group as the reflected Brownian Motion $\{\mathcal{P}_t\}_{t \geq 0}$. Plugging next the explicit formula (9.1.7) for the joint density of $(\mathcal{W}_t, \mathcal{M}_t)$, after the change of variable $z = 2b - a$, we find that

$$Q = \int_{y}^{\infty} db \int_{c+b}^{\infty} \frac{2z}{2\pi u^2} e^{-\frac{z^2}{2u}} dz = \int_{y}^{\infty} \frac{2}{2\pi u} \left[ e^{-\frac{(c+b)^2}{2u}} - e^{-\frac{(c-y+2b)^2}{2u}} \right] db$$

$$= P(\mathcal{N}(-c, u) \geq y) = p_n(0, (y + c, \infty)) = p_n(y, (-\infty, -c)),$$

and thereby complete the proof.

(b). The Markov process $\{Y_t, t \geq 0\}$ has the same initial value and Markov semi-group as the reflected Brownian Motion $\{|\mathcal{W}_t|, t \geq 0\}$, hence these two S.P. have the same law (see Theorem [5.3.2]).

**Solution. 9.1.19**

(a). For fixed $n$ let $\xi_{k,n}$ be i.i.d. each having the law of $T_{b/n}$. Then, by parts (d) and (b) of Exercise 9.1.18

$$n^{-2} \sum_{k=1}^{n} \xi_k \xrightarrow{D} n \sum_{k=1}^{n} (T_{kb/n} - T_{(k-1)b/n}) = T_b,$$

so the law of $T_b$ is indeed $1/2$-stable.

(b). Fixing $y \geq 0$, by the representation of part (a) with i.i.d. $\xi_{k,n}$ and the expression provided in [9.1.2] for $F_{T_{b/n}}(y)$, we get that

$$\log P\left(n^{-2} \max_{1 \leq k \leq n} \xi_k \leq y\right) = n \log P_0(T_{b/n} \leq y) = n \log \left[ 1 - P(\{G\} \leq \frac{b}{n\sqrt{y}}) \right].$$

With $g(t) = P(\{G\} \leq t)$, the limit $n \to \infty$ is calculated by l’Hôpital’s rule to be

$$-\frac{b}{\sqrt{y}} g'(0) = -\frac{b\sqrt{\pi}}{\sqrt{2y}},$$

which proves the result.

To put this in the context of Exercise 3.2.13 note that for $y$ large,

$$P(T_b \leq y) = 1 - P(\{G\} \leq b/\sqrt{y}) \approx 1 - \frac{2b}{\sqrt{2\pi y}}$$

so $P(\pi T_b/(2b^2) \leq y) \approx 1 - y^{-1/2}$. Therefore, by part (b) of Exercise 3.2.13 we expect that $n^{-2} \max_{1 \leq k \leq n} \xi_k \xrightarrow{D} 2b^2 M_\infty / \pi$ where $P(M_\infty \leq y) = \exp(-y^{-1/2})$, as we indeed just proved.

**Additional solutions provided.**

**Solution. 9.1.19**

The Brownian Markov process $(\mathcal{W}_t, \mathcal{F}_t)$ has the Brownian semi-group $p_t(x, \cdot) = \mathcal{P}_x(\mathcal{W}_t \in \cdot)$ of transition probabilities and starts at $W_0 \geq 0$.

(a). Since $p_t(x, \{0\}) = 0$ and $p_t(x, B) \geq p_t(x, -B)$ for any $t, x > 0$ and Borel $B \subseteq (0, \infty)$, it follows that $p_{-t}(\cdot, \cdot)$ is a transition probability on $\mathbb{R}_+$ for each $t > 0$. For such $B$ we have by the definition of $p_{-t}(\cdot, \cdot)$ that $p_{-t}p_{-u}(0, B) = 0$.
and moreover, if \( x > 0 \) then by the symmetry of the Brownian semi-group,
\[
p_{-,s}p_{-,u}(x, B) = \int_{0+}^{\infty} p_{-,s}(x, dy)[p_u(y, B) - p_u(y, -B)]
\]
\[
= \int_0^{\infty} p_s(x, dy)[p_u(y, B) - p_u(y, -B)] - \int_{-\infty}^0 p_s(x, dy)[p_u(-y, B) - p_u(-y, -B)]
\]
\[
= p_s p_u(x, B) - p_s p_u(x, -B) = p_{s+u}(x, B) - p_{s+u}(x, -B) = p_{-,s+u}(x, B).
\]
Considering \( B = (0, \infty) \) and any \( x \geq 0 \), we have in addition that
\[
p_{-,s}p_{-,u}(x, \{0\}) = 1 - p_{-,s}p_{-,u}(x, (0, \infty)) = 1 - p_{-,s+u}(x, (0, \infty)) = p_{-,s+u}(x, \{0\}),
\]
so \( p_{-,t}(\cdot, \cdot) \) is a Markov semi-group on \( \mathbb{R}_+ \).

With \( T_0 = \inf \{ s > 0 : W_s = 0 \} \), by sample path continuity \( W_{T_0} = 0 \) and starting at \( W_0 \geq 0 \), clearly \( X_t := W_{t+T_0} \) is an \( \mathbb{R}_+ \)-valued, \( \mathcal{F}_t \)-adapted process. Consequently, to show that \((X_t, \mathcal{F}_t)\) is a homogeneous Markov process on \( \mathbb{R}_+ \) with the specified transition probabilities, it suffices to show that for any \( u > 0, s \geq 0 \) and \( B \subseteq (0, \infty) \),
\[
\mathbf{P}(X_{s+u} \in B | \mathcal{F}_s) = p_{-,u}(X_s, B)
\]
(for in terms of \( B \) both sides of (*) are probability measures on \( \mathbb{R}_+ \), so upon considering \( B = (0, \infty) \) the identity extends to its complement \( \{0\} \) and thereafter to any Borel subset of \( \mathbb{R}_+ \).

Turning to establish (*), fixing \( u > 0, s \geq 0 \) and \( B \subseteq (0, \infty) \), let \( \mathbf{P}_y \) stand for the law of the Brownian Markov process \((W_t, \mathcal{F}_t)\) starting at \( W_0 = y \). Note that if \( X_{s+u} > 0 \) then \( T_0 > s + u \), hence \( X_{s+u} = W_{s+u} \), \( X_s = W_s > 0 \) and by the Markov property of \((W_t, \mathcal{F}_t)\) at \( t = s \),
\[
\mathbf{P}(X_{s+u} \in B | \mathcal{F}_s) = \mathbf{I}_{\{W_s > 0\}} \mathbf{P}(W_{s+u} \in B, T_0 > s + u | \mathcal{F}_s)
\]
\[
= \mathbf{I}_{\{W_s > 0, T_0 > s\}} \mathbf{P}_{W_s}(W_u \in B, T_0 > u) = \mathbf{I}_{\{X_s > 0\}} \mathbf{P}_{X_s}(W_u \in B, T_0 > u).
\]
Therefore, the claimed identity (*) is merely a consequence of
\[
\mathbf{P}_y(W_u \in B, T_0 > u) = p_{-,u}(y, B)
\]
holding for all \( y > 0 \). But \( \mathbf{P}_y(T_0 = u) = 0 \) and \( p_{-,u}(y, B) = p_u(y, B) - p_u(y, -B) \), so the identity (**) is equivalent to
\[
\mathbf{P}_y(W_u \in B, T_0 < u) = \mathbf{P}(W_u \in y - B, T_y < u) = p_u(y, y + B) = p_u(y, -B),
\]
of which the middle, and only non-trivial, equality is just the ‘extended reflection’ identity \([9.1.6] \) of Exercise \([9.1.12] \) (proved there for any open interval \( B \subseteq (0, \infty) \), from which the general case easily follows by the \( \pi - \lambda \) theorem).

(b). Note that \( p_{+,t}(x, \cdot) = p_t(x, \cdot) + p_t(x, -\cdot) \) is for \( t > 0 \) merely the law of \( |W_t| \) starting at \( W_0 = x \in \mathbb{R} \) (and in particular \( p_{+,t}(\cdot, \cdot) \) is a transition probability on \( \mathbb{R}_+ \), for each \( t > 0 \)). Further, \( p_t(x, B) = p_t(-x, -B) \) and therefore \( p_{+,t}(x, \cdot) = p_{+,t}(|x|, \cdot) \). Consequently, starting at \( W_0 = x \), by the Markov property of \((W_t, \mathcal{F}_t)\), for any \( s \geq 0, u > 0 \) and Borel \( B \subseteq [0, \infty) \),
\[
\mathbf{P}_x(|W_{s+u}| \in B | \mathcal{F}_s) = \mathbf{P}_{W_s}(|W_u| \in B) = p_{+,u}(W_s, B) = p_{+,u}(|W_s|, B).
\]
Taking the expectation of both sides, we get from the tower property that for any \( x \in \mathbb{R} \),
\[
p_{+,s+u}(x, B) = \mathbf{E}_x[\mathbf{P}(|W_{s+u}| \in B | \mathcal{F}_s)] = \mathbf{E}_x[p_{+,u}(|W_s|, B)] = p_{+,s}p_{+,u}(x, B),
\]
so \( p_{+,t}(\cdot, \cdot) \) is indeed a Markov semi-group on \( \mathbb{R}_+ \).
SOLUTION. 9.1.18
(a) Let $b_n \uparrow b$ so that $T_{b_n}(\omega)$ is a non-decreasing sequence, hence has a limit $T'(\omega)$ and our claimed left-continuity of $b \mapsto T_b$ amounts to showing that $T'(\omega) = T_b(\omega)$. 

To this end, note first that since $W_0 = 0 \leq b_n < b$, it follows by continuity of $t \mapsto W_t(\omega)$ that $T_b \geq T_{b_n}$ for all $n$, hence also $T_b \geq T'$. For the converse we assume no loss that $T'(\omega)$ is finite. Hence, continuity of $t \mapsto W_t(\omega)$ at $t = T'(\omega)$ implies that $b_n = W_{T_{b_n}}(\omega) \to W_{T'}(\omega)$, so necessarily $W_{T'}(\omega) = b$ (from which we easily deduce that $T_b(\omega) \leq T'(\omega)$). That $T_b$ is strictly increasing in $b$ follows from the continuity of the Wiener process (for if $T_b = T_b'$ then $b' = W_{T_{b_n}} = W_{T_b} = b$).

From Corollary 9.1.6, $\tau_0 = \inf\{t \geq 0 : W_t < 0\}$ is a.s. equal to zero, so that for any $\epsilon > 0$ there exists $t \in (0, \epsilon)$ with $W_t < 0$. For such $t$, by continuity of the sample functions there exist $t' < t < t''$ such that $W_{[t', t'' \bar{\epsilon}]} < 0$, which means that $T_{M_t} \leq t'$ while $T_b > t''$ for any $b > M_t$. Thus, $b \mapsto T_b$ fails to be continuous at $b = M_t$, and taking $\epsilon \downarrow 0$ we see that $b \mapsto T_b$ is a.s. discontinuous on any interval $[0, \epsilon']$ with $\epsilon' > 0$. By the strong Markov property applied at time $T_b$, we have that $T_{b'}(W_t - T_b(W_t)) = T(\tilde{W})$ for the Brownian Markov process $\tilde{W}_t = W_{T_b + t} - W_{T_b}$. Consequently, considering the preceding for $\tilde{W}$ we deduce that w.p.1. $b \mapsto T_b$ is discontinuous on every interval of the form $[b, b + \epsilon']$ where $b, \epsilon' \in \mathbb{Q}_{>0}$, hence also on any interval in $[0, \infty)$ of non-trivial length.

(b). We prove the stated identities by an application of the monotone class theorem. To this end, fixing $a > 0$, since a.s. $T_b < \infty$ and $W_{T_b} = b$, we have for $W_0 = 0 \leq b < c$ that

$$P(T_c - T_b > a | \mathcal{F}_{T_b}^+) = P(\max_{T_b \leq s \leq T_b + a} W_s < c | \mathcal{F}_{T_b}^+)$$

$$= P(\max_{T_b \leq s \leq T_b + a} \{W_s - W_{T_b}\} < c - b | \mathcal{F}_{T_b}^+) = P(\max_{0 \leq s \leq a} \widetilde{W}_s < c - b | \mathcal{F}_{T_b}^+),$$

where $\{\widetilde{W}_s = W_{s + T_b} - W_{T_b}, s \geq 0\}$ is independent of $\mathcal{F}_{T_b}^+$ (consider Corollary 9.1.6 for the Markov time $T_b$). Hence,

$$P(T_c - T_b > a | \mathcal{F}_{T_b}^+) = P(\widetilde{M}_a < c - b) = P_0(T_{c-b} > a).$$

Moreover, considering $B_s = W_s + b$ adds $b$ to the corresponding running maxima and therefore,

$$P_b(T_c > a) = P_b(M_a < c) = P_0(M_a + b < c) = P_0(T_{c-b} > a).$$

In conclusion, we have shown that

$$E[h(T_c - T_b) | \mathcal{F}_{T_b}^+] = E_0[h(T_c)] = E_0[h(T_{c-b})],$$

for any $h(\cdot) = I_{(a, \infty)}(\cdot)$, with $a > 0$. The $\pi$-system of sets $(a, \infty)$, $a > 0$ generates the Borel $\sigma$-algebra of $\mathbb{R}_+$. Further, the vector space of $h \in bB$ for which (*) hold, contains constants and is invariant under monotone bounded limits, so by the monotone class theorem, (*) holds for all $h \in bB$.

(c). Since $W_0 = 0$ and $t \mapsto W_t$ is continuous, $b \mapsto T_b$ is non-decreasing. Consequently, recall Exercise 8.1.11 that with each $T_b$ being an $\mathcal{F}_b$-stopping time, the canonical filtration $\mathcal{G}_b = \sigma(T_x, 0 \leq x \leq b)$ of $\{T_b, b \geq 0\}$ is contained in $\mathcal{F}_{T_b}^+$ for each $b \geq 0$. Therefore, by part (b), the tower property and the formula (9.1.4) for
the density \( f_{T_\gamma}(\cdot) \), for any \( t, b \geq 0 \) and Borel subset \( B \) of \( \mathbb{R}_+^{+} \),

\[
P\left( T_b + t - T_b \in B \mid \mathcal{G}_t \right) = \mathbb{E}\left[ P\left(T_b + t - T_b \in B \mid \mathcal{F}_{T_b}^+ \right) \mid \mathcal{G}_b \right] = \mathbb{P}_0(T_t \in B) = \int_B \frac{t}{\sqrt{2\pi u^3}} e^{-\frac{t^2}{2u}} du.
\]

That is, \( \{T_b, b \geq 0\} \) is a process of stationary, independent increments, which are non-negative due to the monotonicity of \( b \mapsto T_b \). As such it is a homogeneous Markov process of semi-group \( \tilde{q}_b(x, B) = \mathbb{P}(T_b + t - T_b \in B \mid \{y - x : y \in B\}) \) (see Proposition 8.3.3), whose transition probability kernel \( \tilde{q}_b(x, y) \) we have already computed.

(d). To prove that \( \mathbb{P}(\tau_{b+} = T_b) = 1 \) for all \( b \geq 0 \), recall Corollary 9.1.5 that \( \mathbb{P}(\tau_0+ = 0) = 1 \), and with \( \tilde{W}_s = W_{T_b+s} - W_{T_b} \) as in part (b),

\[
\mathbb{P}_0(\tau_{b+} = T_b) = \mathbb{P}_0(\inf\{s \geq T_b : W_s > b\} = T_b) = \mathbb{P}_0(\inf\{s \geq 0 : \tilde{W}_s > 0\} = 0) = \mathbb{P}_0(\tau_0+ = 0) = 1.
\]

Turning to the right continuity of \( b \mapsto \tau_{b+}(\omega) \), fix non-random \( c_n \downarrow b \) and \( \varepsilon > 0 \). By definition of \( \tau_{b+} \) there exist \( \tau = \tau(\omega) < \tau_{b+} + \varepsilon \) such that \( W_{\tau}(\omega) > b \), hence \( \tau_{c_\alpha} \leq \tau < \tau_{b+} + \varepsilon \) for all \( n \geq n_0(\omega, \varepsilon) \) large enough. Considering \( \varepsilon \downarrow 0 \) we deduce that \( \tau_{c_\alpha} \downarrow \tau_{b+} \), and since this applies for any \( c_n \downarrow b \), conclude that \( \{\tau_{b+}, b \geq 0\} \) is a right-continuous modification of \( \{T_b, b \geq 0\} \). By part (c) and Exercise 8.3.20 these processes have a Feller semi-group, and as such the right continuity of \( b \mapsto \tau_{b+} \) guarantees that it is a strong Markov process (see Proposition 8.3.19).

(e). The density of \( T_\gamma \) is

\[
f_{T_\gamma}(t) = \frac{|c|}{\sqrt{2\pi t^3}} e^{-\frac{t^2}{2c^2}} = \frac{1/c^2}{\sqrt{2\pi(t/c^2)^3}} e^{-\frac{1}{2(c^2)} t^2} = \frac{1}{c^2} f_{T_1}(t/c^2).
\]

Thus, \( T_\gamma \overset{D}{=} c^2 T_1 \).