310c: Homework Solutions 2019

Homework 1

Solution. 7.1.11

(a). This follows by a minor adaptation of the proof of Lemma 7.1.10. Specifically, let

\[ A_i = \{ x \in \mathbb{R}^i \mid s \mapsto x(s) \text{ is linear, with } A_i, \]

\[ i = 2, \ldots, 6 \text{ denoting the corresponding collections of all polynomial, constant, } \]

non-decreasing, differentiable, analytic functions, respectively. If \( A_i \subseteq B^i \) for some \( i \leq 6 \), then by Lemma 7.1.7 there exists a countable \( \{ t_k \} \subseteq I \) such that the values of \( x(t_k) \) determine whether \( x(\cdot) \in A_i \) or not. However, for each \( 1 \leq i \leq 6 \), all \( x \in A_i \) and any \( u \in \mathbb{I} \), there exists \( y \notin A_i \) with \( x(s) = y(s) \) for all \( s \in \mathbb{I} \), \( s \neq u \), so none of these six sets can be in \( B^i \). By the same reasoning, if either the collection \( C_i \) of all \( x \in \mathbb{R}^3 \) that are continuous at some fixed \( t \), is in \( B^i \), or the (linear) subspace \( BV_i \)

of all elements of \( \mathbb{R}^i \) having finite total variation on \( \mathbb{I} \), is in \( B^i \), then there exists some countable \( \{ t_k \} \subseteq \mathbb{I} \) such that \( y \in C_i \) (or \( y \in BV_i \)), whenever \( x(t_k) = y(t_k) \) for some \( x \in C_i \) (or \( x \in BV_i \), respectively), and all \( k \). This is of course false, for there always exists a monotone sequence \( k \mapsto u_k \in \mathbb{I} \), disjoint of \( \{ t_k \} \), such that \( u_k \to t \) (hence \( y \in C_i \) requires also that \( y(u_k) \to y(t) \), and \( y \in BV_i \) requires instead that \( \sum_k |y(u_{k+1}) - y(u_k)| \) is finite).

(b). The collections \( A_2 = \{ x(\cdot) : x(s) = 0 \text{ for some } s \in \mathbb{I} \}, A_3 = \{ x(\cdot) : x(s) < x(t) \}

\text{for some } s < t \in \mathbb{I} \}, \) are not in \( B^i \) for the same reason that \( A_i, i \leq 6 \) are not. Similarly, the set \( M X = \{ x(\cdot) : x(s) \geq x(u) \text{ for some } s < t \in \mathbb{I} \} \) is not in \( B^i \) for the same reason that \( BV_i \) is not.

(c). We apply the same reasoning as in part (a), namely suppose there exists \( A \subseteq C(\mathbb{I}) \) non-empty such that \( A \subseteq B^i \). Then, there exists \( x \in A \) and a countable base \( \{ t_k \} \subseteq \mathbb{I} \) for \( A \). In particular, any \( y \in \mathbb{R}^3 \) such that \( y(t_k) = x(t_k) \) for all \( k \), must be in \( A \), hence also in \( C(\mathbb{I}) \). But, fixing any collection of values for \( y(\cdot) \) on \( \{ t_k \} \)

is not enough to guarantee its continuity throughout \( \mathbb{I} \), yielding a contradiction to our assumption that such \( A \subseteq B^i \) exists.

In contrast, fixing a monotone sequence \( t_k \in \mathbb{I} \) whose limit is \( t_\infty \in \mathbb{I} \), some \( x_\infty \in \mathbb{R} \) and a sequence \( \{ x_k \} \) whose limit as \( k \to \infty \) is not \( x_\infty \), the non-empty set

\[ A = \{ x(\cdot) : x(t_k) = x_k, 1 \leq k \leq \infty \} \]

is obviously in \( B^i \) and contains only functions \( x(s) \) which are discontinuous at \( s = t_\infty \).

(d). We first show that the \( \sigma \)-algebra \( B^i \) contains no non-empty subset \( \Gamma \) of the

set \( A = B(\mathbb{I}) \) of all Borel measurable functions. Indeed, the existence of such \( \Gamma \)

implies (as in part (c)), the existence of countable \( \{(t_k, x_k)\} \subseteq \mathbb{I} \times \mathbb{R} \) such that

any \( y : \mathbb{I} \to \mathbb{R} \) for which \( y(t_k) = x_k, k = 1, 2, \ldots \), must be in \( A \). However, this is not possible, since countable \( \{ t_k \} \subseteq \mathbb{I} \) is a Borel measurable set, so there exists \( T \subseteq \mathbb{I} \setminus \{ t_k \} \) not in the Borel \( \sigma \)-algebra \( B^i \) and hence \( y(s) = x 1_T(s) + \sum_k x_k 1_{t_k}(s) \)

is necessarily not in \( A \) (whenever \( x \notin \{ 0, x_k \} \).

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From the preceding we deduce that if $A \in \mathcal{B}^2$ then $A$ must be a $(\mathcal{P}, \mathcal{B}^2)$-null set, namely, $A \subseteq G \in \mathcal{B}^2$ for some $G$ such that $\mathcal{P}(G) = 0$. It then follows that $F = G^c \in \mathcal{B}^1$ is a non-empty subset of $A^c$. We claim that this can not be the case. Indeed, if such $F$ exists, then by Lemma 4.4.4 there must exist countable $\{t_k, x_k\}$ such that there is no Borel function $x(\cdot)$ on $\mathbb{R}$ for which $x(t_k) = x_k$ for all $k$. In particular, setting $x(s) = x \notin \{x_k\}$ for all $s \notin \{t_k\}$, should result with a non-Borel measurable function. However, since $I \setminus \{t_k\}$ is a Borel subset of $\mathbb{R}$, for $x(\cdot)$ to be non-Borel measurable would have required that some (countable) subset of $\{t_k\}$ is a non-Borel set, which is clearly impossible.

**Solution.** 7.1.12

Fixing $h > 0$ and $t \geq 0$, let $Y = X_{t+h} - X_t$ and $\mathcal{L} = \{A \in \mathcal{F} : \mathcal{P}(A \cap C) = \mathcal{P}(A)\mathcal{P}(C) \text{ for all } C \in \mathcal{Y}\}$. Note that $\mathcal{L}$ is a $\sigma$-system. Also, for $x = (s_1, \ldots, s_m)$ such that $0 \leq s_1 \leq \cdots \leq s_m = t$ considering our assumption that $X_{s_1}, X_{s_2} - X_{s_1}, \cdots, X_{s_m} - X_{s_{m-1}}$ are $\mathcal{P}$-mutually independent, with and without $s_{m+1} = t + h$, we deduce from Definition 1.4.3 that $\{X_{s_1} \in B_1\} \cap_{k=2}^m \{X_{s_k} - X_{s_{k-1}} \in B_k\} \in \mathcal{L}$ for any $(B_1, B_2, \ldots, B_m) \in \mathcal{B}^m$. Since this $\pi$-system generates $\mathcal{F}_x = \sigma(X_{s_1}, X_{s_2} - X_{s_1}, \cdots, X_{s_m} - X_{s_{m-1}})$, by Dynkin’s $\pi$–$\lambda$ theorem we have that $\mathcal{F}_x \subseteq \mathcal{L}$.

The measurable map from $(X_{s_1}, X_{s_2}, \cdots, X_{s_m})$ to $(X_{s_1}, X_{s_2} - X_{s_1}, \cdots, X_{s_m} - X_{s_{m-1}})$ is invertible, hence $\mathcal{F}_x = \sigma(X_{s_1}, X_{s_2}, \cdots, X_{s_m})$ (see Exercise 1.2.33). Let $\mathcal{P} = \cup \mathcal{F}_x$ where the union is over all finite sets $x \subseteq [0,t]$. Since $\mathcal{P}$ is a $\pi$-system, appealing once more to the $\pi$–$\lambda$ theorem you conclude that $\mathcal{F}_x^\mathcal{P} = \sigma(\mathcal{P}) \subseteq \mathcal{L}$. That is, as claimed $X_{t+h} - X_t$ is independent of $\mathcal{F}_x^\mathcal{P}$.

**Solution.** 7.1.13

Given $A_1, \ldots, A_n \in \mathcal{T}$, we define the corresponding f.d.d. as follows. First set $B_{k1} = A_k = B_{k0}^c$ for $k = 1, \ldots, n$ and note that by monotonicity of $A \mapsto \mu(A)$, for each non-zero $b = (b_1, \ldots, b_n) \in \{0,1\}^n$ the set $A_b = \cap_{k=1}^n B_{kb}$ is in $\mathcal{T}$. With $N_k$ a Poisson $(\mu(A_k))$ random variable independent of $(N_{k'}, k' \neq k)$, let the random vector $(N_{A_1}, \ldots, N_{A_n})$ have the same law as $(\sum_{k=1}^n N_{k1}) N_{k1}, k = 1, \ldots, n$.

Then, considering $n = 1$ we see that $N_{A}$ has the Poisson $(\mu(A))$ law for each $A \in \mathcal{T}$. Moreover, if $A_k, k = 1, \ldots, n$ are disjoint sets then $\sum b_k > 1$ yields $A_b = \emptyset$ and hence $N_b = 0$. So, in this case our construction results with mutually independent $N_{A_k}$, $k = 1, \ldots, n$. Similarly, for $A_{n+1} = \cup_{k=1}^n A_k$ and disjoint $A_k, k \leq n$, we have $A_{b}^c$ non-empty only for $b = 0$ or when $b_k = b_{n+1} = 1$ are the only two non-zero coordinates of $b$. Hence, in this case $N_{A_{n+1}} = \sum_{k=1}^n N_{A_k}$, as claimed.

Turning to check that these f.d.d. are consistent, fix $n$ and a permutation $\pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$. Note that for index sets $\{A_{\pi(k)}\}$ the preceding constructs mutually independent $N_{b_{\pi(1)}, \ldots, b_{\pi(n)}}$ of Poisson $(\mu(A_b))$ laws, hence the resulting law of $(N_{A_{\pi(1)}}, \ldots, N_{A_{\pi(n)}})$ is merely the image of the law of $(N_{A_1}, \ldots, N_{A_n})$ under the permutation $\pi$ of the coordinates. That is, the identity (7.1.1) holds (where $A_k \in \mathcal{T}$ serve as the index points $t_k$ there). Next, fixing $\{A_k, k \leq n\}$ and non-zero $b \in \{0,1\}^{n-1}$, since $A_b$ is the disjoint union of $A_{b0}$ and $A_{b1}$ we have by finite additivity of $\mu(\cdot)$ that $\mu(A_b) = \mu(A_{b0}) + \mu(A_{b1})$. This applies for any non-zero $b$, so by the thinning property of Poisson variables we have the representation $N_b = N_{b0} + N_{b1}$ (see Exercise 3.4.10 part (a)), and as a result the identity (7.1.2) also holds.

Finally, by Proposition 7.1.8 there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and on it a stochastic process $A \mapsto N_A$ indexed by $\mathcal{T}$ which has the specified f.d.d.
Applying now the Kolmogorov-Centsov continuity theorem with \(\alpha = 2\) and \(\beta = p - 1\) yields the stated result.

(b). Clearly \(G = X_t - X_s\) is a \(\mathcal{N}(0, v)\) random variable and as in part (a)
\[
v = \mathbf{E}[(X_t - X_s)^2] \leq 2C|t - s|^{p-1},
\]
for some finite constant \(C\) and all \(s, t \in \mathbb{I}\).

Recall that if \(G\) is a \(\mathcal{N}(0, v)\) R.V. then \(\mathbf{E}[G^{2k}] = c_k v^k\) for some universal constants \(c_k\) (for example, from (1.3.18) one easily finds that \(c_k = \frac{(2k)!}{2^k k!}\)). Consequently, for some finite \(C_k\)
\[
\mathbf{E}|X_t - X_s|^{2k} = c_k v^k \leq C_k|t - s|^{(p-1)k}, \quad \forall t, s \in \mathbb{I}.
\]

Applying now the Kolmogorov-Centsov continuity theorem with \(\alpha = 2k, \beta = k(p - 1) - 1\) and \(k = k(\gamma)\) large enough so \(\gamma < (p-1)/2 - 1/(2k)\), we deduce the existence of a locally \(\gamma\)-Hölder continuous modification of \(\{X_t, t \in \mathbb{I}\}\). While this is not enough, by adapting the proof of the Kolmogorov-Centsov continuity theorem, one can actually construct one continuous modification of \(\{X_t, t \in \mathbb{I}\}\) which is locally \(\gamma\)-Hölder for all \(\gamma < (p-1)/2\).

**Solution. 7.2.13**

(a). Clearly \(\sup_{t \in \mathbb{J}} X_t \geq \sup_{s \in \mathbb{C} \cup \mathbb{J}} X_s\). Conversely, for any \(t \in \mathbb{J}\) there exists a sequence \(s_k \to t, s_k \in \mathbb{C}\) such that \(X_t = \lim_{k} X_{s_k}\). Since \(\mathbb{J}\) is open we have \(\lim_{k} X_{s_k} \leq \sup_{s \in \mathbb{C} \cup \mathbb{J}} X_s\), so \(\sup_{t \in \mathbb{J}} X_t = \sup_{s \in \mathbb{C} \cup \mathbb{J}} X_s\). The right-hand side is the supremum of countably many elements of \(m_{\mathcal{F}^X}\) so it belongs also to \(m_{\mathcal{F}^X}\).

(b). Clearly, \(\sup_{t \in (s,s+h)} |X_t - X_s| = \sup_{t \in (s,s+h)} |X_t - X_s|\) and the claim follows by arguing as in (a) for the open interval \(\mathbb{J} = (s,s+h)\).

**Additional solutions provided.**

**Solution. 7.2.3**

Since \(\{X_t\}\) and \(\{Y_t\}\) are modifications of each other, by definition \(\mathbf{P}(N_t) = 0\) for \(N_t = \{\omega : X_t(\omega) \neq Y_t(\omega)\}\) and each \(t \geq 0\). Moreover, up to a null set \(N_s\) such that \(\mathbf{P}(N_s) = 0\) both \(t \mapsto X_t(\omega)\) and \(t \mapsto Y_t(\omega)\) are right-continuous functions.

Therefore,
\[
\bigcup_{t \geq 0} N_t \subseteq N_s \cup \bigcup_{q \in \mathbb{Q}^+} N_q.
\]

Assuming as usual that our probability space is complete, we consequently find that
\[
\mathbf{P}(\exists t \geq 0, X_t \neq Y_t) \leq \mathbf{P}(N_s) + \sum_{q \in \mathbb{Q}^+} \mathbf{P}(N_q) = 0.
\]

That is, \(\{X_t\}\) and \(\{Y_t\}\) are indistinguishable.

**Solution. 7.2.7**
(a). We use induction on \( k \). Starting with \( k = 1 \) we need only consider \( m = 0 \) and show that

\[
\sup_{t,s \in Q_1^{(2,1)}, |t-s| < 1} |x(t) - x(s)| \leq 2\Delta_{1,1}(x) = 2\left[ |x\left(\frac{1}{2}\right) - x(0)| \lor |x(1) - x\left(\frac{1}{2}\right)| \right].
\]

This inequality is trivial, for \( Q_1^{(2,1)} = \{0, \frac{1}{2}, 1\} \).

Next, assuming the stated inequality holds for \( k - 1 \) and any \( k - 1 > m \geq 0 \), we define for each \( s < t \in Q_1^{(2,k)} \)

\[
s' = \min\{u \in Q_1^{(2,k-1)} : u \geq s\}, \quad t' = \max\{u \in Q_1^{(2,k-1)} : u \leq t\}.
\]

Note that \( s', t' \in Q_1^{(2,k-1)} \) are such that \( s \leq s' \leq t' \leq t \), \( s' - s \leq 2^{-k} \) and \( t - t' \leq 2^{-k} \). Hence,

\[
|x(t) - x(t')| \leq \Delta_{k,1}(x), \quad |x(s') - x(s)| \leq \Delta_{k,1}(x).
\]

Further, if \( t - s < 2^{-m} \) for \( m = k - 1 \) then \( t' = s' \) so \( |x(t') - x(s')| = 0 \), whereas if \( t - s < 2^{-m} \) only for some \( m < k - 1 \), then by the induction hypothesis and the fact that \( t' - s' \leq t - s \),

\[
|x(t') - x(s')| \leq 2 \sum_{\ell=m+1}^{k-1} \Delta_{\ell,1}(x).
\]

Combining these inequalities we find that as claimed

\[
|x(t) - x(s)| \leq |x(t) - x(t')| + |x(t') - x(s')| + |x(s') - x(s)| \\
\leq \Delta_{k,1}(x) + 2 \sum_{\ell=m+1}^{k-1} \Delta_{\ell,1}(x) + \Delta_{k,1}(x) = 2 \sum_{\ell=m+1}^{k} \Delta_{\ell,1}(x).
\]

(b). For any pair \( s, t \in Q_1^{(2)} \) satisfying \( 0 < |t - s| < 2^{-m} \) we have that \( 2^{-(m+1)} \leq |t - s| < 2^{-m} \) for some \( m \geq n \), and \( s, t \in Q_1^{(2,k)} \) for all \( k \) large enough. Thus, by part (a) and our assumption that \( \Delta_{\ell,1}(x) \leq 2^{-\gamma \ell} \),

\[
|x(t) - x(s)| \leq \sup_{u, v \in Q_1^{(2,k)}, |u - v| < 2^{-m}} |x(u) - x(v)| \\
\leq 2 \sum_{\ell=m+1}^{\infty} \Delta_{\ell,1}(x) \leq 2 \sum_{\ell=m+1}^{\infty} 2^{-\gamma \ell} = c_\gamma 2^{-\gamma (m+1)} \leq c_\gamma |t - s|^{\gamma}.
\]

**Solution.** Let \( \mathbb{T} = [0, \infty) \), recall that the topology induced on \( C(\mathbb{T}) \) by uniform convergence on compact subsets of \( \mathbb{T} \) is equivalent to that of the separable metric \( \mathbb{S} = (C(\mathbb{T}), \rho(\cdot, \cdot)) \), where \( \rho(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi(||x - y||_j) \) and \( \varphi(r) = r/(1 + r) \). With \( \mathcal{C} = \{A \subseteq C(\mathbb{T}) : A \in \mathcal{B}^2\} \), upon following the proof of Lemma 7.2.8 we have that \( \mathcal{B}_0 = \mathcal{C} \) provided we show that any open ball \( B(x, r) = \{y \in C(\mathbb{T}) : \rho(x, y) < r\} \) of \( \mathbb{S} \) is in \( \mathcal{C} \). To this end, recall that in the course of proving Lemma 7.2.8 we have shown that the sets \( B(x, r, m) = \{y \in C(\mathbb{T}) : ||y - x||_m < r\} \) are in the \( \sigma \)-algebra \( \mathcal{C} \) for each \( x \in C(\mathbb{T}) \), \( r > 0 \) and finite \( m \), hence it suffices to prove the following representation

\[
B(x, r) = \bigcup_{n \geq 1} \bigcup_{g \in \Gamma_n(r_n)} \bigcap_{j=1}^{n} B(x, q_j, j),
\]
in terms of countable unions/intersections of such sets, where \( r_n = r - 2^{-n} \) and for each positive integer \( m \),

\[
\Gamma_m(r) = \{ q \in \mathbb{Q}_+^m : \sum_{j=1}^m 2^{-j} \varphi(q_j) < r \}.
\]

Considering the sequence \( d_j = \| y - x \|_j \), convince yourself that this representation is a direct consequence of the identity

\[
\{ d \in \mathbb{R}_+^\infty : f_{\infty}(d) < r \} = \bigcup_{n \geq 1} \{ d \in \mathbb{R}_+^\infty : f_n(d) < r_n \},
\]

where \( f_{\infty}(d) = \sum_{j=1}^\infty 2^{-j} \varphi(d_j) \), \( f_n(d) = \inf \{ \sum_{j=1}^n 2^{-j} \varphi(q_j) : q_j > d_j, q_j \in \mathbb{Q}_+ \} \). To verify this identity note that \( f_n(d) = \sum_{j=1}^n 2^{-j} \varphi(d_j) \) because of the continuity of \( \varphi : \mathbb{R}_+ \rightarrow [0, 1] \). As \( \sup d \sum_{j=n}^\infty 2^{-j} \varphi(d_j) \leq 2^{-n} \), if \( f_n(d) < r_n \) then necessarily \( f_{\infty}(d) < r \), whereas conversely \( f_{\infty}(d) < r \) implies that \( f_n(d) \leq f_{\infty}(d) < r_n \) for all \( n \) large enough.
Homework 2

Solution. 7.3.4 This exercise is stated and proved as [Dud89 Theorem 12.1.3]. (a) By Definition 7.3.1 if \( X_t \) is a Gaussian process defined on an index set \( T \) then the joint distribution of \( (X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \) is a multivariate normal \( \mathcal{N}(\mu, \Sigma) \) for all \( t_k \in T, k = 1, 2, \ldots, n \) and \( n < \infty \). By Proposition 7.3.14 such distribution is in turn uniquely determined by the vector \( \mu = (m(t_1), m(t_2), \ldots, m(t_n)) \) and the matrix \( \Sigma = (c(t_j, t_k))_{j,k=1,2,\ldots,n} \). Thus, the f.d.d. of such a process (which by Proposition 7.1.8 uniquely determine its law), are unambiguously defined by the corresponding mean function \( m(\cdot) \) and auto-covariance function \( c(\cdot, \cdot) \).

(b) As in part (a) the mean and auto-covariance functions specify the f.d.d. \( \nu_{t_1, t_2, \ldots, t_n} \) to be \( \mathcal{N}(\mu, \Sigma) \) which is well defined since the non-negative definiteness of the auto-covariance function, per condition (7.3.1), implies that the corresponding matrix \( V \) must also be non-negative definite. In view of our canonical construction (see Proposition 7.1.8), it thus suffices to show that these f.d.d. are consistent.

To this end, first note that Definition 7.3.13 of \( \nu_{t_1, t_2, \ldots, t_n} = \mathcal{N}(\mu, \Sigma) \) via its characteristic function is such that for any \( B_k \in \mathcal{B}, k = 1, \ldots, n \) and permutation \( \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \)
\[
\nu_{t_1, t_2, \ldots, t_n}(B_1 \times B_2 \times \cdots \times B_n) = \nu_{\pi(1), \pi(2), \ldots, \pi(n)}(B_{\pi(1)} \times B_{\pi(2)} \times \cdots \times B_{\pi(n)}).
\]

This is the consistency condition (7.1.1). Next note that the multivariate normal characteristic functions \( \Phi_{t_1, t_2, \ldots, t_n}(\cdot) \) corresponding to the f.d.d. \( \nu_{t_1, t_2, \ldots, t_n}(\cdot) \) are such that for any \( t_k \in T \) and \( \underline{a} \in \mathbb{R}^{n-1} \),
\[
\Phi_{t_1, \ldots, t_{n-1}}(\underline{a}) = \Phi_{t_1, \ldots, t_n}((\underline{a}, 0))
\]

With \( X \in \mathbb{R}^n \) and \( X' \in \mathbb{R}^{n-1} \) denoting random vectors such that \( \Phi_X(\cdot) = \Phi_{t_1, \ldots, t_n}(\cdot) \) and \( \Phi_{X'}(\cdot) = \Phi_{t_1, \ldots, t_{n-1}}(\cdot) \), it is easy to see that such relation holds if \( X' \) consists of the first \( n-1 \) coordinates of \( X \) and as each characteristic function uniquely determines the law of the corresponding random vector, the f.d.d. \( \nu_{t_1, t_2, \ldots, t_n}(\cdot) \) must satisfy also the consistency condition (7.1.2).

Solution. 7.3.10

(a) Note that for a weakly stationary, process \( \{X_s, s \geq 0\} \) and any \( t, h \geq 0 \),
\[
0 \leq \text{Var}(X_{t+h} + X_t) = c(t+h, t+h) + c(t, t) + 2c(t, t + h) = 2[r(0) + r(h)].
\]

Consequently, \( |r(h)| \leq r(0) \) for all \( h \geq 0 \). Further, if \( r(h) = r(0) \) then by the preceding \( \text{Var}(X_{t+h} - X_t) = 0 \) for each \( t \geq 0 \), and consequently \( X_{t+h} \overset{d}{=} X_t \).

(b) When such process \( \{X_s, s \geq 0\} \) has in addition independent increments, it follows that
\[
r(h) - r(0) = c(t, t + h) + c(t, t) = \text{Cov}(X_{t+h} - X_t, X_t) = 0.
\]

This applies for all \( h \geq 0 \), so by part (a) we deduce that \( \{X_s, s \geq 0\} \) satisfies the condition (7.2.1) of Kolmogorov-Centsov theorem (with \( \alpha = 2, c = 0 \) and any \( \beta > 0 \)). Hence, this S.P. has a continuous modification. Further, w.p.1., by part (a) also \( X_h(\omega) = X_0(\omega) \) for all \( h \in \mathbb{Q}, h \geq 0 \). In particular, replacing \( \{X_s\} \) by its continuous modification, the latter identity extends to all \( h \geq 0 \) and \( \omega \in \Omega \), as claimed.

Solution. 7.3.13

(a) Fixing \( n \) and \( 0 = s_0 \leq s_1 < \cdots < s_n \), let \( a_j = \sqrt{s_j - s_{j-1}}, j \geq 1 \) and \( \{G_j\} \) be i.i.d. standard random variables. Then, \( (B_{s_1}, \ldots, B_{s_n}) \) have the distribution of
(S_1, \ldots, S_n), where S_0 = x and S_m = S_0 + \sum_{j=1}^{m} a_j G_j. These are clearly multivariate normal distributions (as can be directly verified, or alternatively deduced from Exercise 3.5.20). Further, deleting a point s_k in this construction merely replaces in each of the sums S_m, m = k + 1, \ldots, n the term Y = a_k G_k + a_{k+1} G_{k+1} by Y' = a' G', where a' = \sqrt{s_{k+1} - s_k} and G' is another standard normal variable independent of \{G_j\}. Since a'^2 = a_k^2 + a_{k+1}^2, it follows that Y \overset{D}{=} Y' (see Lemma 3.1.1), and the resulting joint law is thus not affected by this change. That is, the f.d.d. satisfy the condition (7.1.3) and as such are consistent (see Lemma 7.1.4).

Considering the f.d.d. for s_1 = s, n = 1, clearly \( \mathbb{E} B_s = x \) and \( \mathbb{E}(B_s - x)^2 = s \) for any \( s \geq 0 \). Moreover, considering the f.d.d. for \( n = 2 \) and \( 0 \leq s_1 < s_2 \), by the independence of \( B_{s_2} - B_s \), and \( B_{s_1} - x \)
\[
c(s_1, s_2) = \text{Cov}(B_{s_1}, B_{s_2}) = \mathbb{E}[(B_{s_1} - x)(B_{s_2} - x)] = \mathbb{E}[(B_{s_1} - x)^2] = s_1.
\]
Hence, \( c(s, t) = s \land t \), as required.

We note in passing that in view of Exercise 7.3.4 about the consistency of Gaussian f.d.d. one may replace the explicit construction provided here by a direct analytic proof that the function \( c(s, t) = s \land t \) is non-negative definite.

(b). From the representation we used in part (a) we have that for each \( 0 \leq s_1 < s_2 \) the R.V. \( B_{s_2} - B_{s_1} \) follows the \( \mathcal{N}(0, s_2 - s_1) \) law. Thus, in particular
\[
\mathbb{E}[(B_t - B_s)^4] = 3(\mathbb{E}[(B_t - B_s)^2])^2 = (t - s)^2
\]
(c.f. our solution for part (b) of Exercise 7.2.13), and setting \( I = [0, T] \), the existence of a continuous modification \( \{W_t, t \in I\} \) follows upon applying the Kolmogorov-Centsov continuity theorem (with \( \alpha = 4 \) and \( \beta = 1 \)).

(c). As summarized in Corollary 7.2.16 the continuous modification \( \{W_t, t \in I\} \) of the S.P. \( \{B_t, t \in I\} \) is a measurable mapping from \( (\Omega, \mathcal{F}) \) to the Borel \( \sigma \)-algebra of the separable metric space \( (C(I), \| \cdot \|_{\infty}) \). Further, as in our solution for part (b) of Exercise 7.2.13 note that for \( k \geq 1 \), some universal finite constants \( c_k \) and all \( t, s \geq 0 \),
\[
\mathbb{E}[(B_t - B_s)^{2k}] = c_k |t - s|^k.
\]
Therefore, fixing \( \gamma < 1/2 \) and considering Kolmogorov-Centsov’s theorem with \( \alpha = 2k \) large enough yields the existence of a \( \gamma \)-locally Hölder continuous modification of \( \{B_t, t \in I\} \) (as in the solution of Exercise 7.2.13). Since any two continuous modifications of the same S.P. are indistinguishable, w.l.o.g. the same modification \( \{W_t, t \in I\} \) is identical to \( \gamma \)-locally Hölder continuous for all \( \gamma < 1/2 \).

(d). With \( \mathbb{E}B_t^2 = t \) depending on \( t \), this process is not even weakly stationary. However, by bi-linearity of the covariance,
\[
\text{Cov}(B_t - B_u, B_s) = c(t, s) - c(u, s) = t \land s - u \land s = 0
\]
for every \( 0 \leq s \leq u < t \). As \( \{B_t, t \geq 0\} \) is a continuous time, Gaussian S.P., from Corollary 7.3.6 we have that \( \{B_t, t \geq 0\} \) has independent increments. Further, as shown in part (b), the R.V. \( B_t - B_s \) follows the \( \mathcal{N}(0, |t - s|) \) law, hence by Definition 7.3.11 the S.P. \( \{B_t, t \geq 0\} \) has stationary increments.

**Solution.** 7.3.16

(a). Suppose first that \( s > t \). Then \( W_s - W_t \) having the \( \mathcal{N}(0, s - t) \) law, is independent of \( W_t \). Therefore, by linearity of the expectation \( \mathbb{E}[W_s | W_t] = W_t \) and \( \text{Var}[W_s | W_t] = \text{Var}[W_s - W_t] = \mathbb{E}[(W_s - W_t)^2] = s - t \).
Moving to deal with $s < t$, note that $(W_s, W_t)$ has the $\mathcal{N}(0, \mathbf{V})$ law for the invertible two-dimensional covariance matrix $\mathbf{V} = \begin{bmatrix} s & s \\ s & t \end{bmatrix}$. Computing $\mathbf{V}^{-1}$ we thus arrive at the joint probability density function

$$f_{W_s, W_t}(x, y) = \exp(-\frac{x^2}{2s} - \frac{(y-x)^2}{2(t-s)}/(2\pi \sqrt{s(t-s)})].$$

With the density of $W_t$ being $f_{W_t}(y) = \exp(-y^2/(2t))/\sqrt{2\pi t}$, we find that the conditional density of $W_s$ given $W_t$ is $f_{W_s|W_t}(x|W_t)$, where

$$f_{W_s|W_t}(x|y) = \frac{f_{W_s, W_t}(x, y)}{f_{W_t}(y)} = \exp(-\frac{(x-y)^2}{2v})/(\sqrt{2\pi v})$$

and $v = s(t-s)/t$. As the latter is the density of the $\mathcal{N}(sy/t, v)$ law, we conclude that $\mathbf{E}[W_s|W_t] = (s/t)W_t$ and $\text{Var}[W_s|W_t] = s(t-s)/t$.

(b). Since $\{W_t, t \geq 0\}$ is a MG of continuous sample functions, we know from Doob's $L^2$ maximal inequality that for $I_n = [2^{n-1}, 2^n]$ and any $n \geq 1$,

$$\mathbf{E}\left[\sup_{s \in I_n} W_s^2\right] \leq 4\mathbf{E}[W_2^2] = 2^{n+2}.$$  

Hence, for any $\varepsilon > 0$, by Markov’s inequality we deduce that

$$\varepsilon^2 \mathbf{P}\left(\sup_{s \in I_n} |s^{-1} W_s| \geq \varepsilon\right) \leq \mathbf{E}\left[\sup_{s \in I_n} |s^{-1} W_s|^2\right] \leq 2^{-2(n-1)} \mathbf{E}\left[\sup_{s \in I_n} W_s^2\right] \leq 2^{4-n}.$$  

Since these bounds are summable in $n$, by Borel-Cantelli I we have that for each $\varepsilon > 0$, w.p.1. $\sup_{s \in I_n} |s^{-1} W_s| < \varepsilon$ for all $n$ large enough. Considering $\varepsilon \downarrow 0$ we then conclude that $s^{-1} W_s \overset{a.s.}{\rightarrow} 0$ when $s \to \infty$.

(c). Considering $t \in \mathbb{I} = [0, 1]$, the f.d.d. of both $\hat{B}_t = W_t - tW_1$ and $\tilde{B}_t = (1-t)W_t/(1-t)$ (with $\hat{B}_1 = 0$), are laws of finite linear combinations of coordinates of certain Gaussian random vectors, hence in either case follow a multivariate normal distribution (see Exercise 3.3.20). Clearly, $\mathbf{E}[\hat{B}_t] = 0 = \mathbf{E}[\tilde{B}_t]$ for all $t \in \mathbb{I}$. Thus, $\{B_t, t \in \mathbb{I}\}$ and $\{\hat{B}_t, t \in \mathbb{I}\}$ are two Gaussian processes of zero mean functions, hence have the same law provided their auto-covariance functions coincide (see Exercise 7.3.3). Using bi-linearity of the covariance, we easily calculate these functions, as follows

$$\text{Cov}(\tilde{B}_s, \tilde{B}_t) = \text{Cov}\left((1-s)W_s/(1-s), (1-t)W_t/(1-t)\right) = (1-s)(1-t)(4s(t-s)/(1-s)^2) = s(t-s) = s \wedge t - st;$$

$$\text{Cov}(\tilde{B}_s, \hat{B}_t) = \text{Cov}(W_s - sW_1, W_t - tW_1) = \text{Cov}(W_s, W_t) - s \text{Cov}(W_s, W_1) - t \text{Cov}(W_1, W_s) + st \text{Cov}(W_1, W_1) = s \wedge t - st - ts + st = s \wedge t - st.$$  

In conclusion, as claimed $\{\hat{B}_t, t \in \mathbb{I}\} \overset{d}{=\sim} \{\tilde{B}_t, t \in \mathbb{I}\}$.

The continuity of $t \mapsto \hat{B}_t$ on $[0, 1)$ follows from the sample path continuity of the Wiener process $\{W_s, s \geq 0\}$. Further, setting $s = \frac{1-t}{1-t'}$, we see that $\hat{B}_{t'} = \frac{t'}{1-t'} \left(\frac{W_s}{s}\right)$, so from part (b) we deduce that $\hat{B}_{t'} \overset{a.s.}{\rightarrow} 0$ when $t' \uparrow 1$. We thus conclude that w.p.1. the sample functions of $\{\hat{B}_t, t \in \mathbb{I}\}$ are continuous.

(d). Let $\{B'_t, t \in \mathbb{I}\}$ denote the S.P. specified by the f.d.d. of $\{W_t, t \in \mathbb{I}\}$, conditioned upon $W_1 = 0$. The latter are conditional marginals of a multivariate normal distribution and hence themselves multivariate normal. Thus, $\{B'_t, t \in \mathbb{I}\}$ is a
Gaussian S.P. whose mean function is zero by part (a). So, as in part (c), it suffices to show that $\text{Cov}(B_s', B_t') = s - t$. The case $s = t$ is covered by part (a), so without loss of generality assume hereafter that $s < t$. Now, recall that $(W_s, W_t, W_1)$ follows a multivariate normal distribution of covariance matrix
\[
\begin{bmatrix}
s & s & s \\
 s & t & t \\
 s & t & 1 \\
\end{bmatrix}.
\]

Since $((W_s, W_t)|W_1 = 0)$ is the conditional marginal of this distribution on its first two coordinates, it has the covariance matrix
\[
\begin{bmatrix}
s & s & s \\
 s & t & t \\
 s & t & 1 \\
\end{bmatrix} - \begin{bmatrix}
s \\
 s \\
 s \\
\end{bmatrix} [1]^{-1} \begin{bmatrix}
s & s \\
 s & t \\
\end{bmatrix} = \begin{bmatrix}
s & s & s \\
 s & t & t \\
 s & t & 1 \\
\end{bmatrix} - \begin{bmatrix}
s^2 & st \\
 st & t^2 \\
\end{bmatrix}.
\]

We conclude that $\text{Cov}(B_s', B_t') = s - t$ for any $s \leq t \in \mathbb{I}$, so $\{B_t', t \in \mathbb{I}\}$ has the same distribution as the standard Brownian bridge.

**Solution.**

(a). If $\tau$ is a Markov time, then $\{\tau \leq t - \epsilon\} \in \mathcal{F}_{(t-\epsilon)+} \subseteq \mathcal{F}_t$ for all $t$ and $\epsilon > 0$, hence $\{\tau < t\} = \bigcup_n \{\tau \leq t - \frac{1}{n}\} \in \mathcal{F}_t$ for all $t$. Conversely, if $\{\tau < t\} \in \mathcal{F}_t$ for all $t$, then $\{\tau \leq t\} = \bigcup_n \{\tau < t + \frac{1}{n}\} \in \mathcal{F}_t + \mathbb{Q} = \mathcal{F}_t$', hence $\tau$ is a Markov time.

(b). If $\{\tau_n, n \in \mathbb{Z}_+\}$ are $\mathcal{F}_t$-stopping times, then $\{\tau_1 \wedge \tau_2 \leq t\} = \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathcal{F}_t$, so $\tau_1 \wedge \tau_2$ is an $\mathcal{F}_t$-stopping time, and $\{\sup_n \tau_n \leq t\} = \bigcap_n \{\tau_n \leq t\} \in \mathcal{F}_t$, so $\sup_n \tau_n$ is also an $\mathcal{F}_t$-stopping time.

Given $A = \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\}$, the event $\tau_1 + \tau_2 > t$ occurs if and only if the open interval $(t - \tau_1, \tau_2) \subseteq (0, t)$ is nonempty, thus contains some rational point $q \in (0, t)$. Hence, $A \cap \{\tau_1 + \tau_2 > t\} = A \cap B$, where
\[
B = \bigcup_{q \in \mathbb{Q} \cap (0, t)} \{\tau_2 > q\} \cap \{\tau_1 > t - q\}.
\]

Since each set in the latter countable union belongs to $\sigma(\mathcal{F}_q, \mathcal{F}_{t-q}) \subseteq \mathcal{F}_t$, it follows that $A \cap \{\tau_1 + \tau_2 > t\} \in \mathcal{F}_t$. Further, for $i = 1, 2$ we have that $\{\tau_i > t\} \cap \{\tau_1 + \tau_2 > t\} = \{\tau_i > t\} \in \mathcal{F}_t$. Considering the union of these events we deduce that $\{\tau_1 + \tau_2 > t\} \in \mathcal{F}_t$, hence $\{\tau_1 + \tau_2 \leq t\} = \{\tau_1 + \tau_2 > t\}^c \in \mathcal{F}_t$. This applies for all $t \geq 0$, so $\tau_1 + \tau_2$ is an $\mathcal{F}_t$-stopping time.

(c). Since $\mathcal{F}_t$-Markov times are merely $\mathcal{F}_t^+$-stopping times, in case $\{\tau_n, n \in \mathbb{Z}_+\}$ are $\mathcal{F}_t$-Markov times, applying part (b) for the filtration $\{\mathcal{F}_t^+, t \geq 0\}$ we have that $\tau_1 + \tau_2$ and $\sup_n \tau_n$ are $\mathcal{F}_t$-Markov times. Moreover by part (a), $\{\inf_n \tau_n < t\} = \bigcup_n \{\tau_n < t\} \in \mathcal{F}_t$ for all $t \geq 0$, hence $\inf_n \tau_n$ is also an $\mathcal{F}_t$-Markov time. Finally,
\[
\lim_{n \to \infty} \tau_n = \inf_{n \geq n^*} \tau_k \quad & \quad \lim_{n \to \infty} \sup_{n \geq k} \tau_k = \inf_{n \to \infty} \tau_k,
\]
hence these two random variables are also $\mathcal{F}_t$-Markov times.

(d). Suppose $\tau_1$ and $\tau_2$ are $\mathcal{F}_t$-Markov times. Fixing $t \geq 0$, recall part (a) that $\tilde{A} = \{\tau_1 < t\} \cap \{\tau_2 < t\} \in \mathcal{F}_t$. Further, for any $0 < q < t$ both $\{\tau_2 > q\} \in \mathcal{F}_q \subseteq \mathcal{F}_t$ and $\{\tau_1 > t - q\} \in \mathcal{F}_{(t-q)+} \subseteq \mathcal{F}_t$, hence the set $B$ of part (b) is in $\mathcal{F}_t$ and by the same argument as in part (b), $\tilde{A} \cap \{\tau_1 + \tau_2 > t\} = \tilde{A} \cap B$ is also in $\mathcal{F}_t$.

Now, if both $\tau_1$ and $\tau_2$ are strictly positive (for all $\omega \in \Omega$), then $\tilde{A}^c = \{\tau_1 \geq t\} \cup \{\tau_2 \geq t\} \in \mathcal{F}_t$ is contained in $\{\tau_1 + \tau_2 > t\}$. Consequently, in this case, for any $t \geq 0$,
\[
\{\tau_1 + \tau_2 > t\} = \tilde{A}^c \cup (\tilde{A} \cap B) \in \mathcal{F}_t.
\]
That is, \( \tau_1 + \tau_2 \) is an \( \mathcal{F}_t \)-stopping time.

Alternatively, if \( \tau_1 \) is also an \( \mathcal{F}_t \)-stopping time, then \( \tilde{A} = \{ \tau_1 \leq t \} \cap \{ \tau_2 < t \} \in \mathcal{F}_t \) and as in the preceding \( \tilde{A} \cap \{ \tau_1 + \tau_2 > t \} = \tilde{A} \cap B \in \mathcal{F}_t \). Assuming in addition that \( \tau_1 > 0 \) for all \( \omega \in \Omega \), results with \( \tilde{A}^c = \{ \tau_1 > t \} \cup \{ \tau_2 \geq t \} \) which is contained in \( \{ \tau_1 + \tau_2 > t \} \). Hence, also in this case \( \tau_1 + \tau_2 \) is an \( \mathcal{F}_t \)-stopping time.

**Additional solutions provided.**

**Solution. 7.3.3**

By definition, the weak stationarity of a (square-integrable) continuous time S.P. \( X_t \) amounts to having constant mean function \( m(t) = E X_t \) and an auto-correlation function of the form \( c(t, s) = r(|t - s|) \). In particular, for each \( n \) and \( t_1, \ldots, t_n \) both the vector \( \mu = (m(t_1 + s), \ldots, m(t_n + s)) \) and the matrix \( \mathbf{V} = (c(t_j + s, t_k + s))_{j,k=1,\ldots,n} \) are independent of \( s \geq 0 \). Assuming that \( X_t \) is further a Gaussian process, from part (a) of Exercise 7.3.4 we have that the multivariate normal distribution of \( (X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \) is also independent of \( s \geq 0 \). That is, the S.P. \( X_t \) satisfies the equivalent formulation (7.3.2) of strict stationarity.

For a weakly stationary process which is not stationary, try \( Z_t = (-1)^{N_t} Z_0 \) where \( N_t \) is a Poisson process of rate one which is independent of \( Z_0 \), a bounded variable such that \( E Z_0 = 0 \) and \( E Z_0^3 = 1 \). Indeed, by independence \( m(t) = E Z_t = E(-1)^{N_t} E Z_0 = 0 \) and since \((-1)^2 = 1\), for any \( t \geq s \),

\[
c(t,s) = E Z_t Z_s = E Z_0^3 E(-1)^{N_t-N_s} = r(|t-s|),
\]

because \( N_t-N_s \overset{\text{d}}{=} N_{t-s} \) whose law depends only on \( t-s \). Consequently, \( Z_t \) is weakly stationary. However, with \( N_t \) a Poisson(\( t \)) variable, we have by independence that

\[
E Z_t^3 = E Z_0^3 E(-1)^{3N_t} = \sum_{k=0}^{\infty} \frac{(-1)^3 t^k}{k!} e^{-t} = e^{-2t}.
\]

With \( E Z_t^3 \) depending on \( t \) the S.P. \( Z_t \) can not be stationary.

Another example of this type, taken from [GS01] Example 8.2.5, is the weakly stationary \( Y_t = A \cos(pt/2) + B \sin(pt/2) \), where \( A, B \) are uncorrelated but not identically distributed variables of zero-mean and unit variance. This process can not be strictly stationary since the laws of \( Y_0 = A \) and \( Y_1 = B \) are not the same.

**Solution. 7.3.17**

(a). Using hereafter \( \| \cdot \| \) for \( \| \cdot \|_2 \), clearly for any \( t \geq s \geq 0 \) and \( x \in \mathbb{R} \),

\[
g_t(s+x) = \frac{|t|^{H-1/2} g(|s+x|/t)}{\|g\|} = \frac{1}{\|g\|} \left[ |t-s-x|^{H-1/2} \text{sgn}(t-s-x) + |s+x|^{H-1/2} \text{sgn}(s+x) \right].
\]

Consequently, simple algebra yields that

\[
g_t(s+x) - g_s(s+x) = \frac{1}{\|g\|} \left[ |t-s-x|^{H-1/2} \text{sgn}(t-s-x) + |x|^{H-1/2} \text{sgn}(x) \right] = g_{t-s}(x).
\]

The latter identity obviously applies also for \( s = 0 \) and implies that \( \|g_t - g_s\| = \|g_{t-s}\| \) for all \( t \geq s \geq 0 \). Further, one easily checks that, from its definition, \( \|g_t\|^2 = |t|^{2H} \) for any \( t \geq 0 \). Therefore, for any \( t \geq s \geq 0 \),

\[
2c(t,s) = \|g_t\|^2 + \|g_s\|^2 - \|g_{t-s}\|^2 = \|g_t\|^2 + \|g_s\|^2 - \|g_t - g_s\|^2 = 2 \int g_t(x) g_s(x) dx.
\]

(b). In view of part (a) we have from part (b) of Exercise 7.3.4 that existence of fBM requires only the non-negative definiteness of \( c(t,s) = \int g_t(x) g_s(x) dx \). This
non-negative definiteness is obvious, as for any \(a_k \in \mathbb{R}, t_k \geq 0, k = 1, \ldots, n\) and finite \(n\),
\[
\sum_{j=1}^{n} \sum_{k=1}^{n} a_j c(t_j, t_k) a_k = \int \left( \sum_{j=1}^{n} a_j g_{t_j}(x) \right) \left( \sum_{k=1}^{n} a_k g_{t_k}(x) \right) dx \geq 0
\]
(compare with (7.3.1)).

Next, from the given expression for \(c(t, s)\) we deduce that the fBM \(\{X_t, t \geq 0\}\) of parameter \(H\) satisfies the identity
\[
\mathbb{E}[(X_t - X_s)^2] = c(t, t) + c(s, s) - 2c(t, s) = |t - s|^{2H}.
\]
So, now follow the line of reasoning we have used in solving part (c) of Exercise 7.3.13. That is, as in our solution for part (b) of Exercise 7.2.13, since the continuous time S.P. \(X_t\) is Gaussian, we have that \(\mathbb{E}[(X_t - X_s)^{2k}] = c_k |t - s|^{2Hk}\) for any \(k \geq 1\), some universal finite constants \(c_k\) and all \(t, s \geq 0\). Hence, fixing \(\gamma < H\) and considering Kolmogorov-Centsov’s theorem with \(\alpha = 2k\) large enough (and \(\beta = 2Hk - 1\)), yields the existence of a \(\gamma\)-locally Hölder continuous modification of \(\{X_t, t \geq 0\}\) (and as in the solution of Exercise 7.2.13, the same modification can be taken to be \(\gamma\)-locally Hölder continuous for all \(0 < \gamma < H\)).

(c). The auto-covariance function of fBM with parameter \(H = 1/2\) is by definition \(c(t, s) = (t + s - |t - s|)/2 = t \wedge s\). The continuous modification of this S.P. is further centered, Gaussian and of continuous sample functions, hence it is a standard Wiener process (see Definition 7.3.12).

(d). Fixing non-random \(b > 0\), since the fBM \(\{X_t, t \geq 0\}\) is a centered, Gaussian S.P. the same applies also for \(Y_t = b^{-H}X_{bt}\). Further, clearly,
\[
\mathbb{E}(Y_t Y_s) = b^{-2H} c(bt, bs) = c(t, s)
\]
for any \(t, s \geq 0\), namely the auto-covariance of \(\{Y_t, t \geq 0\}\) matches that of the fBM. We thus conclude that the two processes have the same law, as stated.

(e). In part (b) we saw that for any \(t \geq s \geq 0\) the increment \(X_t - X_s\) of fBM is a Gaussian R.V. of zero-mean and variance \(|t - s|^{2H}\). In particular, its law depend only on \(t - s\) which by Definition 7.3.11 implies that the fBM is a S.P. of stationary increments (for any \(0 < H < 1\)).

Next, a Gaussian S.P. has independent increments if and only if its auto-covariance is of the form \(c(t, s) = h(t \wedge s)\) (c.f. the remark just after Corollary 7.3.6). In case of fBM this amounts to \(c(s + u, s) - c(s, s) = (s + u)^{2H} - u^{2H}\) independent of \(u \geq 0\), which holds if and only if \(H = 1/2\).
Solution. 8.1.11

(a) It is easy to verify that for any $\sigma$-algebras $\mathcal{H} \subseteq \mathcal{G}$ and any $H \in \mathcal{H}$, the collection
$$\mathcal{H}^H = \{A \in \mathcal{G} : A \cap H \in \mathcal{H}\}$$
is a $\sigma$-algebra (see part (b) of Exercise 1.1.13). Since $\tau$ is an $\mathcal{F}_t$-stopping time, for any $t \geq 0$ fixed, $H_t = \{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \subseteq \mathcal{F}_\infty$, so $(\mathcal{F}_t)^H_t$ is a $\sigma$-algebra. Per Definition 8.1.9, $\mathcal{F}_\tau = \bigcap_{t \geq 0}(\mathcal{F}_t)^H_t$ must also be a $\sigma$-algebra.

Recall Definition 1.2.12 that $\sigma(\tau) = \sigma(H_s, s \geq 0)$. Further, $H_s \cap H_t = H_{s \wedge t} \in \mathcal{F}_{s \wedge t} \subseteq \mathcal{F}_t$ for all $s, t \geq 0$. As $H_s \in (\mathcal{F}_t)^H_t$ for all $t \geq 0$, clearly $H_s \in \mathcal{F}_\tau$ regardless of the value of $s \geq 0$. With $\mathcal{F}_\tau$ a $\sigma$-algebra, it then follows that $\sigma(\tau) \subseteq \mathcal{F}_\tau$.

Suppose next that $\tau(\omega) = t_0$ is non-random. Then, $H_t = \Omega$ when $t \geq t_0$ while $H_t = \emptyset$ when $t < t_0$. Since for $\mathcal{G} = \mathcal{F}_\infty$ we have here $(\mathcal{F}_t)^{t_0} = \mathcal{F}_t$ and $(\mathcal{F}_t)^0 = \mathcal{F}_\infty$, it follows that $\mathcal{F}_\tau = \bigcap_{t \geq t_0}(\mathcal{F}_t) = \mathcal{F}_{t_0}$, as claimed.

(b) It is not hard to check that $\mathcal{H}^{H_t \cap H_t} = \mathcal{H}^{H_t} \cap \mathcal{H}^{H_t}$ for any pair $H_t, H_t \in \mathcal{H}$ (see part (c) of Exercise 1.1.13). Setting $H_t = \{\omega : \theta(\omega) \leq t\}$ and $\tilde{H}_t = \{\omega : \theta(\omega) \wedge \tau(\omega) \leq t\}$, we clearly have $\tilde{H}_t = H_t \cup H_t$, hence $(\mathcal{F}_t)^{\tilde{H}_t} = (\mathcal{F}_t)^{H_t} \cap (\mathcal{F}_t)^{H_t}$ for all $t \geq 0$. Considering the intersections over $t \geq 0$ we conclude that $\mathcal{F}_{\theta \wedge \tau} = \mathcal{F}_\theta \cap \mathcal{F}_\tau$.

Next, note that $\theta(\omega) < \tau(\omega)$ if and only if $\theta(\omega) \leq \tau(\omega)$ for some $q \in \mathbb{Q}^{(2)}$, and if in addition $\theta(\omega) \leq t$ then it suffices to consider $q \in \mathbb{Q}^{(2)} := \mathbb{Q}^{(2)} \cup \{t\}$. Hence, for any $t \geq 0$,
$$B_t := \{\omega : \theta(\omega) < \tau(\omega), \theta(\omega) \leq t\} = \bigcup_{q \in \mathbb{Q}^{(2)}} A_q,$$
where $A_q = \{\omega : \theta(\omega) \leq s < \tau(\omega)\}$. Clearly, $A_s = \tilde{H}_s \cap H_s^c \in \mathcal{F}_s \subseteq \mathcal{F}_t$ for any $s \leq t$ and consequently $B_t \in \mathcal{F}_t$ for any $t \geq 0$. Observing that $B_t = \{\theta < \tau\} \cap H_t$ we deduce that $\{\theta < \tau\} \subseteq \mathcal{F}_{\theta \wedge \tau}$.

Finally, by symmetry, $\{\theta \leq \tau\} = \{\tau \leq \theta\} = \{\theta \leq \tau\} \cup \{\theta < \tau\} \subseteq \mathcal{F}_{\theta \wedge \tau}$, and therefore $\{\theta = \tau\} = \{\theta \leq \tau\} \setminus \{\theta < \tau\} \subseteq \mathcal{F}_{\theta \wedge \tau}$.

(c) From part (b) we know that $\{\theta \leq \tau\} \in \mathcal{F}_{\theta \wedge \tau} \subseteq \mathcal{F}_\theta$. So, if $A \in \mathcal{F}_\theta$ then $A \cap H_t = A \cap \tilde{H}_t \subseteq \mathcal{F}_t$. Since $A \cap H_t = \tilde{A} \cap H_t$ we further deduce that $A \cap H_t \subseteq \mathcal{F}_t$ for all $t \geq 0$. That is, $\tilde{A} \in \mathcal{F}_{\theta \wedge \tau}$. Consequently, fixing $Z$ integrable we have by the definition of the C.E. $W = E[Z|\mathcal{F}_{\theta \wedge \tau}]$ that
$$E[Z I_{\theta \leq \tau} I_A] = E[Z I_\tilde{A}] = E[W I_\tilde{A}] = E[W I_{\theta \leq \tau} I_A].$$

This applies for any $A \in \mathcal{F}_\theta$, so by definition $W I_{\theta \leq \tau} \in \mathcal{F}_\theta$ is merely the C.E. $E[Z|\mathcal{F}_\theta]I_{\theta \leq \tau}$ and ‘taking out the known’ $I_{\theta \leq \tau}$ we arrive at the stated identity
$$E[Z|\mathcal{F}_\theta]I_{\theta \leq \tau} = E[Z|\mathcal{F}_{\theta \wedge \tau}]I_{\theta \leq \tau}. \quad (*)$$

Turning to prove our second claim, consider the expected value of $(*)$ conditional on $\mathcal{F}_\tau$. Since $\mathcal{F}_{\theta \wedge \tau} \subseteq \mathcal{F}_\theta$, upon taking in what is known we deduce that
$$E[E[Z|\mathcal{F}_\theta]|\mathcal{F}_\tau]I_{\theta \leq \tau} = E[E[Z|\mathcal{F}_\theta]|\mathcal{F}_\tau] = E[Z|\mathcal{F}_{\theta \wedge \tau}I_{\theta \leq \tau}].$$
Next, interchanging the roles of $\theta$ and $\tau$ in $(*)$ then replacing $Z$ there by the integrable $Z' = E(Z|\mathcal{F}_\theta)$ we further have by the tower property that
$$E[Z'|\mathcal{F}_\theta]I_{\tau \leq \theta} = E[Z'|\mathcal{F}_{\theta \wedge \tau}]I_{\tau \leq \theta} = E[Z|\mathcal{F}_{\theta \wedge \tau}]I_{\tau \leq \theta}. $$
Combining the two identities, we conclude that $E[E(Z|\mathcal{F}_\theta)|\mathcal{F}_\tau] - E[Z|\mathcal{F}_{\theta \wedge \tau}] = 0$ a.s. on the set $\{\theta \leq \tau\} \cup \{\tau \leq \theta\}$. 

Homework 3
(d) Since \( \{ \xi \leq t \} \in \mathcal{F}_\theta \) and \( \theta \leq \xi \) it follows that \( \{ \xi \leq t \} = \{ \xi \leq t \} \cap \{ \theta \leq t \} \in \mathcal{F}_t \) for all \( t \geq 0 \). Hence, \( \xi \) is an \( \mathcal{F}_t \)-stopping time.

**Solution.** 8.2.6

(a). Set \( \langle X \rangle_t = \mathbf{E}X_t^2 - \mathbf{E}X_0^2 \) and fix \( t > s \). Since the zero mean \( X_t - X_s \) is independent of \( \mathcal{F}_s \), it follows that \( \mathbf{E}[X_s(X_t - X_s)|\mathcal{F}_s] = 0 \) as well as also \( \mathbf{E}[X_s(X_t - X_s)] = 0 \) and

\[
\mathbf{E}[X_t^2 - \langle X \rangle_t|\mathcal{F}_s] = \mathbf{E}[X_t^2 + 2X_s(X_t - X_s) + (X_t - X_s)^2|\mathcal{F}_s] - (\mathbf{E}X_t^2 - \mathbf{E}X_0^2)
\]

\[
= X_s^2 + \mathbf{E}[X_t^2 - 2X_s(X_t - X_s)] - (\mathbf{E}X_t^2 - \mathbf{E}X_0^2)
\]

\[
= X_s^2 - \mathbf{E}X_s^2 + \mathbf{E}X_0^2 = X_s^2 - \langle X \rangle_s .
\]

Consequently, \( \langle X_t^2 - \langle X \rangle_t, \mathcal{F}_s \rangle \) is a MG.

(b). Suppose \( \{ X_t, t \geq 0 \} \) is a martingale. Then, by definition \( \mathbf{E}[X_t - X_s|\mathcal{F}_s] = 0 \) for any \( t \geq s \). In particular, \( \mathbf{E}(X_t - X_s) = 0 \). Further, as a Gaussian process \( \{ X_t, t \geq 0 \} \) is square integrable, hence by the tower property and taking out what is known, for any \( u \leq s \)

\[
\mathbf{E}[X_u(X_t - X_s)] = \mathbf{E}[X_u \mathbf{E}(X_t - X_s|\mathcal{F}_s)] = 0 .
\]

Fixing \( 0 \leq t_1 < \cdots < t_n < \infty \), it thus follows that the random vector \( \mathbf{Y}_k = (Y_1, \ldots, Y_n) \) composed of \( Y_1 = X_{t_1} \) and \( Y_k = X_{t_k} - X_{t_{k-1}} \), \( k \geq 2 \), has uncorrelated coordinates. Since \( \mathbf{Y} \) is a Gaussian random vector, by definition 8.2.6 and Proposition 8.5.14 its characteristic function \( \Phi_{\mathbf{Y}}(\mathbf{\theta}) \) is merely the product of the corresponding functions \( \Phi_{Y_k}(\theta_k) \) for the coordinates of \( \mathbf{Y} \), which are thus mutually independent (see Exercise 8.5.11). This in turn implies that \( \{ X_t, t \geq 0 \} \) has zero-mean independent increments (see Exercise 7.1.12), so the conclusion of part (a) applies.

(c). By part (a) and our assumption that \( X_0 = 0 \) we have that \( \langle X_t^2 - \mathbf{E}X_t^2, t \geq 0 \rangle \) is a MG. Further, by stationarity of the zero-mean independent increments of \( \{ X_t, t \geq 0 \} \), for any \( t \geq 0 \) and \( n \geq 1 \),

\[
\mathbf{E}X_t^2 = \mathbf{E}[\langle \sum_{k=1}^n X_{kt/n} - X_{(k-1)t/n} \rangle^2] = \sum_{k=1}^n \text{Var}(X_{kt/n} - X_{(k-1)t/n}) = n\mathbf{E}X_t^{2/n} .
\]

Comparing this identity for \( t = q \in \mathbb{Q}^{(2)} \) and \( n = 2^\ell \) with the one for \( t = 1 \) and \( n = 2^\ell \), we obtain that \( \mathbf{E}X_t^2 = \mathbf{g} \mathbf{E}X_t^2 \) for any \( q \in \mathbb{Q}^{(2)} \). By the monotonicity of \( t \mapsto \mathbf{E}X_t^2 \) (see Exercise 8.2.7), upon considering dyadic rationals \( q_n \) and \( r_n \), such that \( q_n \uparrow s \in \mathbb{R}_+ \) and \( r_n \downarrow s \) we deduce that

\[
s\mathbf{E}X_1^2 = \sup_n \mathbf{E}X_n^2 \leq \mathbf{E}X_s^2 \leq \inf_n \mathbf{E}X_n^2 = s\mathbf{E}X_1^2 .
\]

Therefore, \( \mathbf{E}X_s^2 = s\mathbf{E}X_1^2 \) for all \( s \geq 0 \), which concludes the proof.

**Solution.** 8.2.7

(a). Adaptedness is clear, and the integrability of \( u_0(t, B_t, \theta) \) follows from the fact that \( B_t \) is \( \mathcal{N}(0, t) \) distributed, hence its moment generating function \( M_t(\theta) := \mathbf{E}[\exp(\theta B_t)] = \exp(\theta^2 t/2) \) is finite for any \( \theta \in \mathbb{R} \). Since the Brownian motion has stationary independent increments, we also have

\[
\mathbf{E}[u_0(t, B_t, \theta)|\mathcal{F}_s] = \mathbf{E}[e^{\theta B_t + \theta(B_t - B_s) - \theta^2 t/2}|\mathcal{F}_s] = e^\theta B_s - \theta^2 t/2 \mathbf{E}[e^{\theta(B_t - B_s)}] = e^\theta B_s - \theta^2 t/2 M_{t-s}(\theta) = e^\theta B_s - \theta^2 t/2 e^{\theta^2 (t-s)/2} = u_0(s, B_s, \theta).
\]
Hence, \( u_0(t,B_t,\theta) \) is an \( F^B_t \) martingale.

(b). With \( u_0(t,y,\theta) = \exp(\theta y - \theta^2 y/2) \) let \( a_{k,r}(\theta) = (-t/2)^r(y - \theta t)^{k-2r}u_0(t,y,\theta) \) and \( c_{k,r} = k!/(k-2r)!r! \). We shall prove the identity

\[
u_k(t,y,\theta) = \frac{\partial^k}{\partial \theta^k} u_0(t,y,\theta) = \sum_{r=0}^{[k/2]} c_{k,r} a_{k,r}(\theta)\]

by induction on \( k \in \mathbb{Z}_+ \). Indeed, it trivially holds for \( k = 0 \) since \( a_{0,0}(\theta) = u_0(t,y,\theta) \) and we have only to consider \( r = 0 \) in the sum. Next note that \( \frac{\partial u_0}{\partial \theta} = (y - \theta t)u_0 \), hence for \( k \geq 1 \),

\[
\frac{d}{d\theta} a_{k-1,r} = (k - 1 - 2r)(-t)a_{k-2,r} + (y - \theta t)a_{k-1,r} = 2(k - 1 - 2r)a_{k-1,r+1} + a_{k,r}.
\]

Moreover, if \( r \leq m = \lfloor (k-1)/2 \rfloor \) then \( c_{k-1,r} = (1 - 2r/k)c_{k,r} \) and for \( s = r + 1 \) we have that \( (k - 1 - 2r)c_{k-1,r} = (s/k)c_{k,s}1_{r<(k-1)/2} \). Consequently, with \( m' = \lfloor k/2 \rfloor \) we find that

\[
\frac{d}{d\theta} \sum_{r=0}^{m} c_{k-1,r}a_{k-1,r} = 2 \sum_{r=0}^{m} (k - 1 - 2r)c_{k-1,r}a_{k-1,r+1} + \sum_{r=0}^{m} c_{k-1,r}a_{k,r} = (k,m') \sum_{r=0}^{m'} c_{k,r} a_{k,r}
\]

since \( m' = m \) unless \( k \) is even in which case \( m < (k - 1)/2 \) and \( m' = m + 1 = k/2 \).

In conclusion, if the stated identity holds for \( k - 1 \geq 0 \) then it holds also for \( k \), since by the preceding

\[
u_k(t,y,\theta) = \frac{\partial}{\partial \theta} u_{k-1}(t,y,\theta) = \frac{d}{d\theta} \sum_{r=0}^{m} c_{k-1,r} a_{k-1,r} = \sum_{r=0}^{[k/2]} c_{k,r} a_{k,r}(\theta) .
\]

By induction, this identity holds for all \( k \in \mathbb{Z}_+ \) and fixing \( \theta = 0 \) gives our claim that

\[
u_k(t,y,0) = \sum_{r=0}^{[k/2]} \frac{k!}{(k-2r)!r!} (-t/2)^r y^{k-2r}.
\]

(c). By definition \( u_{k-1,h}(t,y,\theta) = h^{-1}(u_{k-1}(t,y,\theta + h) - u_{k-1}(t,y,\theta)) \) converges to \( u_k(t,y,\theta) \) when \( h \to 0 \) and by the mean-value theorem

\[
\sup_{|h| \leq 1} |u_{k-1,h}(t,y,\theta)| \leq \sup_{|\eta - \theta| \leq 1} |u_k(t,y,\eta)| =: U_k(t,y,\theta).
\]

In the sequel we show that \( \mathbb{E} U_k(t,B_t,\theta) \) is finite for any \( t, \theta \) and \( k \). It then follows by linearity and dominated convergence of C.E. that w.p.1. for all \( \ell \geq 1 \),

\[
\mathbb{E} [u_\ell(t,B_t,\theta)|F^B_s] = \lim_{h \to 0} h^{-1} \left( \mathbb{E} [u_{\ell-1}(t,B_t,\theta + h)|F^B_s] - \mathbb{E} [u_{\ell-1}(t,B_t,\theta)|F^B_s] \right) = \frac{\partial}{\partial \theta} \mathbb{E} [u_{\ell-1}(t,B_t,\theta)|F^B_s] .
\]

Iterating this identity for \( \ell = k, k - 1, \ldots, 1 \), we deduce from part (a) that w.p.1.

\[
\mathbb{E} [u_k(t,B_t,\theta)|F^B_s] = \frac{\partial^k}{\partial \theta^k} \mathbb{E} [u_0(t,B_t,\theta)|F^B_s] = \frac{\partial^k}{\partial \theta^k} u_0(s,B_s,\theta) = u_k(s,B_s,\theta).
\]
Integrability follows from the same argument for exchanging the order of expectation and differentiation, so \( u_k(t, B_t, \theta) \) is an \( \mathcal{F}_t^B \)-martingale.

As for the integrability of \( U_k(t, B_t, \theta) \), first note that for all \( k \geq 1 \) and some \( c_k \) finite,

\[
|u_k(t, y, \theta)| \leq c_k (|y - \theta|^2 + t/2)^{k/2} u_0(t, y, \theta)
\]

(for example, from the identity we derived in part (b) we easily see that this applies with \( c_k = \max_{\gamma, \epsilon} c_{k,\gamma} \)). Therefore, for some \( C = C(k, t, |\theta|) \) finite,

\[
U_k(t, y, \theta) \leq c_k (|y| + (|\theta| + 1)|t|^2 + t/2)^{k/2} e^{c|\theta|(|\theta|+1)} \leq Ce^{c(|\theta|+2)}
\]

and this clearly yields the integrability of \( U_k(t, B_t, \theta) \).

Finally, evaluating \( c_k,0 = 1, c_k,1 = k(k-1), c_k,2 = k(k-1)(k-2)/2 \) and \( c_{6,3} = 120 \), the following are \( \mathcal{F}_t^B \)-martingales:

\[
\begin{align*}
    u_2(t, B_t, 0) &= c_{2,0}B_t^2 + c_{2,1}(\frac{-t}{2}) = B_t^2 - t; \\
    u_3(t, B_t, 0) &= c_{3,0}B_t^3 + c_{3,1}(\frac{-t}{2})B_t = B_t^3 - 3tB_t; \\
    u_4(t, B_t, 0) &= c_{4,0}B_t^4 + c_{4,1}(\frac{-t}{2})B_t^2 + c_{4,2}(\frac{-t}{2})^2 = B_t^4 - 6tB_t^2 + 3t^2; \\
    u_6(t, B_t, 0) &= c_{6,0}B_t^6 + c_{6,1}(\frac{-t}{2})B_t^4 + c_{6,2}(\frac{-t}{2})^2B_t^2 + c_{6,3}(\frac{-t}{2})^3 \\
    &= B_t^6 - 15tB_t^4 + 45t^2B_t^2 - 15t^3.
\end{align*}
\]

(d). For any \( \theta \in \mathbb{R} \),

\[
\left( \frac{\partial}{\partial t} + \frac{\partial^2}{2 \partial y^2} \right) u_0(t, y, \theta) = -\frac{\partial^2}{2} u_0(t, y, \theta) + \frac{1}{2} \theta^2 u_0(t, y, \theta) = 0.
\]

In particular, since this expression is zero for all \( \theta \), its \( k \)-th partial derivative with respect to \( \theta \) is also zero, for any \( k \geq 1 \). The function \( u_0(\cdot) \) is the exponential of a trivariate polynomial in \( (t, y, \theta) \), and as such it is real-analytic on \( \mathbb{R}^3 \). This means that its partial derivatives commute, hence

\[
\left( \frac{\partial}{\partial t} + \frac{\partial^2}{2 \partial y^2} \right) u_k(t, y, \theta) = \frac{\partial^k}{\partial \theta^k} \left( \left( \frac{\partial}{\partial t} + \frac{\partial^2}{2 \partial y^2} \right) u_0(t, y, \theta) \right) = 0.
\]

That is, \( u_k(t, y, \theta) \) solves the heat equation, for any fixed \( \theta \in \mathbb{R} \) and \( k \geq 0 \).

**Solution.**

8.2.10

(a). For \( 0 \leq s \leq t \) and \( A \in \mathcal{F}_s \), by the martingale property of \( (Z_t, \mathcal{F}_t) \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) we have that

\[
    Q_s(A) = E_{\mathbb{P}}[Z_s I_A] = E[Z_s I_A] = E[Z_t I_A] = E_{\mathbb{P}}[Z_t I_A] = Q_t(A).
\]

Since this applies for any \( A \in \mathcal{F}_s \) we conclude that \( Q_s = Q_t \big|_{\mathcal{F}_s} \).

(b). Fixing \( 0 \leq u \leq s \leq t \) and \( Y \in L^1(\Omega, \mathcal{F}_s, Q_s) \), for any \( A \in \mathcal{F}_u \) we have by part (a) that

\[
\begin{align*}
    E_{Q_t}[Y I_A] &= E_{Q_s}[Y I_A] = E_{P_s}[Y I_A Z_s] = E[Y I_A Z_s] = E[I_A E[Y Z_s | \mathcal{F}_u]] \\
    &= E_{P_u}[I_A E[Y Z_s | \mathcal{F}_u]] = E_{Q_u}[I_A E[Y Z_s | \mathcal{F}_u]] = E_{Q_s}[I_A E[Y Z_s | \mathcal{F}_u]] = E_{Q_u}[I_A E[Y Z_s | \mathcal{F}_u]],
\end{align*}
\]
where the last equality is due to the fact that $Z_u \in m\mathcal{F}_u$ and the identity $Q_u = Q_{1 \mid \mathcal{F}_u}$ of part (a). With the preceding holding for all $A \in \mathcal{F}_u$, it follows by the definition of C.E. that $Q_{t \text{-a.s.}}$

$$E_{Q_t}[Y \mid \mathcal{F}_u] = \frac{E[YZ_u \mid \mathcal{F}_u]}{Z_u}.$$ 

The latter identity also holds $P_{t \text{-a.s.}}$ since the Radon-Nikodym derivative $Z_t = dQ_t / dP_t$ is assumed to be strictly positive. Further, with $P_t = P_{\mid \mathcal{F}_t}$, the same applies $P_{t \text{-a.s.}}$.

(c). It is easily checked that the moment generating function of a Poisson($\lambda$) R.V. $X$ is $m_{\lambda}(\theta) = E[e^{\theta X}] = e^{\lambda(e^\theta - 1)} < \infty$ for any $\theta \in \mathbb{R}$. In particular, the strictly positive process $(Z_t, t \geq 0)$ is integrable. Further, for any $0 \leq u \leq s$, with $N_s - N_u$ having the Poisson($\lambda(s - u)$) law, independently of $\mathcal{F}_u$, we find that for all $\theta \in \mathbb{R}$,

$$Z_u^{-1}E[e^{\theta(N_s-N_u)}Z_s \mid \mathcal{F}_u] = e^{(\lambda-\tilde{\lambda})(s-u)}E[(\tilde{\lambda}/\lambda)e^{\theta(N_s-N_u)} \mid \mathcal{F}_u]$$

$$= e^{(\lambda-\tilde{\lambda})(s-u)}m_{\lambda(s-u)}[\theta + \log(\tilde{\lambda}/\lambda)] = e^{\tilde{\lambda}(s-u)(e^\theta - 1)} = m_{\tilde{\lambda}(s-u)}(\theta).$$

Noting that $m_{\tilde{\lambda}(s-u)}(0) = 1$ we have in particular that $E[Z_s \mid \mathcal{F}_u] = Z_u$, namely, that $(Z_t, \mathcal{F}_t)_{t \geq 0}$ is a strictly positive martingale, with $E[Z_0] = 1$ (since $N_0 = 0$). It then further follows from part (b) that for any $0 \leq u \leq s \leq T$ and all $\theta \in \mathbb{R}$,

$$E_{Q_T}[e^{\theta(N_s-N_u)} \mid \mathcal{F}_u] = Z_u^{-1}E[e^{\theta(N_s-N_u)}Z_s \mid \mathcal{F}_u] = m_{\tilde{\lambda}(s-u)}(\theta).$$

That is, under $Q_T$ the increment $N_s - N_u$ follows the Poisson law of parameter $\tilde{\lambda}(s - u)$ independently of $\mathcal{F}_u$. Since this applies for any $0 \leq u \leq s \leq T$, we conclude that $(N_t, t \in [0, T])$ is a Poisson process of rate $\tilde{\lambda}$ under the measure $Q_T$, for any $T < \infty$.

**Solution.** [S.2.2.1]

Let $\{X_t, t \geq 0\}$ be a uniformly integrable right-continuous sub-martingale with $M_t = \sup_{0 \leq s \leq t} X_s$. Fixing $y > 0$, let $A_t = \{M_t > y\}$ and recall [S.2.3] that

$$P(A_t) \leq y^{-1}E[X_tI_{A_t}].$$

Since U.I. implies $L^1$-boundedness, Doob’s convergence theorem applies, so there exists $X_\infty \in L^1$ such that $X_t \xrightarrow{a.s.} X_\infty$. Further, $A_t \uparrow A_\infty$ hence $X_tI_{A_t} \xrightarrow{a.s.} X_\infty I_{A_\infty}$, and by U.I. of $\{X_t, t \geq 0\}$ this convergence also holds in $L^1$. So, taking $t \to \infty$ in the preceding inequality we find that

$$P(M_\infty > y) = P(A_\infty) \leq y^{-1}E[X_\infty I_{A_\infty}] = y^{-1}E[X_\infty I_{M_\infty > y}],$$

and upon considering $y \uparrow x > 0$, we conclude that $P(M_\infty \geq x) \leq x^{-1}E[X_\infty I_{M_\infty \geq x}]$. Finally, it holds trivially that $E[X_\infty I_{\{M_\infty \geq x\}}] \leq E[(X_\infty)_+]$.

**Additional solutions provided.**

**Solution.** [S.3.1.12]

(a). By definition $A \in \mathcal{F}_\infty$ is in $\mathcal{F}_{t^+}$ if and only if $A \cap \{\tau \leq s\} \in \mathcal{F}_{s+\varepsilon}$ for all $s \geq 0$ and $\varepsilon > 0$. Alternatively, setting $t = s + \varepsilon$ this amounts to

$$A \cap \{\tau \leq t - \varepsilon\} \in \mathcal{F}_t \quad \forall t, \varepsilon > 0 \quad (\ast).$$

We thus merely need to show that $(\ast)$ is equivalent to

$$A \cap \{\tau < t\} \in \mathcal{F}_t \quad \forall t \geq 0 \quad (\ast \ast).$$
Indeed, if (⋆) holds then with \( \{ \tau < t - 1/n \} \) and \( \mathcal{F}_t \) being a \( \sigma \)-algebra, we get (⋆⋆). Conversely, assuming (⋆⋆) holds, with \( s \mapsto \mathcal{F}_s \) non-decreasing we also have that \( A \cap \{ \tau < t - q \} \in \mathcal{F}_t \) for all \( q > 0 \) and since
\[
\{ \tau \leq t - \epsilon \} = \bigcap_{q \in \mathbb{Q}, q < \epsilon} \{ \tau < t - q \},
\]
we get that (⋆) holds as well.

(b). Recall that \( \tau \leq \tau_1 \) and \( \tau < \tau_1 \) whenever \( \tau < \infty \). In particular, \( \{ \tau_1 \leq t \} \subseteq \{ \tau < t \} \) for each fixed non-random \( t \geq 0 \), and hence for any \( A \in \mathcal{F}_{\tau^+} \), setting \( B := A \cap \{ \tau < t \} \in \mathcal{F}_t \) we have that
\[
A \cap \{ \tau_1 \leq t \} = B \cap \{ \tau_1 \leq t \} \in \mathcal{F}_t
\]
(since \( \tau_1 \) is an \( \mathcal{F}_t \)-stopping time). This applies for all \( t \geq 0 \) and consequently, any such \( A \in \mathcal{F}_{\tau^+} \) is also in \( \mathcal{F}_{\tau_1} \), as claimed.

(c). Since \( \{ \tau < t \} = \bigcup_{n} \{ \tau_n < t \} \), if \( A \in \bigcap_{n} \mathcal{F}_{\tau_n^+} \) then by part (a) for each non-random \( t \geq 0 \),
\[
A \cap \{ \tau < t \} = \bigcup_{n} [A \cap \{ \tau_n < t \}] \in \mathcal{F}_t.
\]
By part (c) of Exercise 8.1.10 \( \tau \) is an \( \mathcal{F}_t \)-Markov time and so again by part (a) we see that any such \( A \in \bigcap_{n} \mathcal{F}_{\tau_n^+} \) is also in \( \mathcal{F}_{\tau^+} \). Conversely, applying part (b) of Exercise 8.1.11 for the \( \mathcal{F}_{\tau^+} \)-stopping times \( \tau \) and \( \theta = \tau_n \) (so \( \tau \wedge \theta = \tau \)), we deduce that \( \mathcal{F}_{\tau^+} \subseteq \mathcal{F}_{\tau_n^+} \) for all \( n \) and conclude that as claimed \( \mathcal{F}_{\tau^+} = \bigcap_{n} \mathcal{F}_{\tau_n^+} \).

If in addition \( \tau_n \) is an \( \mathcal{F}_t \)-stopping time and \( \tau < \tau_n \) whenever \( \tau \) is finite, then from part (b) we further have that \( \mathcal{F}_{\tau^+} \subseteq \mathcal{F}_{\tau_n} \). When this applies for all \( n \), we thus deduce that \( \mathcal{F}_{\tau^+} \subseteq \bigcap_{n} \mathcal{F}_{\tau_n} \) and our claim that in this case \( \mathcal{F}_{\tau^+} = \bigcap_{n} \mathcal{F}_{\tau_n} \) is a direct consequence of the preceding identity.
### Homework 4

**Solution.** [S.2.30]

(a) Fixing $u > 0$, recall Corollary [S.2.29] that $X_t^u := X_{t \land u}$ is a right-continuous $\mathcal{F}_t$-submartingale. Further, by the submartingale property $\mathbb{E}[X_u | \mathcal{F}_t] \geq X_u^u$ for all $t \geq 0$. Hence, $X_u^u = X_u \in m\mathcal{F}_\infty$ is the last element of $\{X_t^u, t \geq 0\}$ (see Definition [S.2.23]). Therefore, applying Corollary [S.2.29] for this sub-MG and the $\mathcal{F}_t$-stopping theorem to the MG $\{X_t^u, t \geq 0\}$ we see that

$$\mathbb{E}[X_u | \mathcal{F}_\theta] = \mathbb{E}[X_u^\theta | \mathcal{F}_\theta] \geq X_\theta^\theta = X_{u \land \theta}$$

(with equality in case of a MG).

(b) Recall Corollary [S.2.29] that $X_t^r := X_{t \land r}$ is a right-continuous $\mathcal{F}_t$-submartingale. We assume here that $\{X_t^r, t \geq 0\}$ is U.I. so by Proposition [S.2.23] it has an integrable last element $X_\infty^\infty = \lim_t X_t^r$. By definition $X_\infty^\infty = X_r$ and applying Theorem [S.2.29] for the sub-MG $\{X_t^r, t \geq 0\}$ we further deduce that $X_\theta = X_\theta^\theta$ is integrable. Further, applying Corollary [S.2.27] for this sub-MG we see that

$$\mathbb{E}[X_r | \mathcal{F}_\theta] = \mathbb{E}[X_r^\theta | \mathcal{F}_\theta] \geq X_\theta^\theta = X_\theta.$$

**Solution.** [S.2.33]

Let $\tau = \inf\{t \geq 0 : Z_t \geq x\}$. Since the $\mathcal{F}_t$-adapted process $Z_t$ has continuous sample path and $[x, \infty)$ is a closed set, it follows from Proposition [S.1.14] that $\tau$ is an $\mathcal{F}_t$-stopping time. Consequently, $Z_\tau = Z_{t \land \tau}$ is a right-continuous $\mathcal{F}_t$-MG (by Corollary [S.2.29]). Further, $0 \leq Z_t^r \leq x$ it is certainly U.I. and hence has a last element $Z_\infty^\infty = Z_\tau \in L^1$ (by Proposition [S.2.23] c.f. solution of part (b) of Exercise [S.2.30]). The sample path continuity of $Z_t$ and our assumption that $Z_0 = 1 < x$, $Z_t \to 0$ as $t \to \infty$ imply that $Z_\infty^\infty = x_{t \land \infty}$. Consequently, applying Doob’s optional stopping theorem to the MG $\{Z_t^r, t \geq 0\}$ we find that

$$1 = \mathbb{E}[Z_0] = \mathbb{E}[Z_0^r] = \mathbb{E}[Z_\infty] = xP(\tau < \infty) = xP(\sup_{t \geq 0} Z_t \geq x).$$

**Warning:** To see that $Z_t \to 0$ is needed for our conclusion, simply consider the martingale $Z_t \equiv 1$ for which $P(\sup_{t \geq 0} Z_t \geq x) = 0$ as soon as $x > 1$. Similarly, if $P(Z_0 > 1) > 0$ then our formula for $P(\sup_{t \geq 0} Z_t \geq x)$ has to be modified to $x^{-1} \mathbb{E}[Z_0 \land x]$.

**Solution.** [S.2.35]

(a) Fixing $b > 0$ and $r \in \mathbb{R}$ let $X_t := Z_t^{(r)} = W_t + rt$ and $\eta := \tau_b^{(r)} = \inf\{t \geq 0 : X_t \geq b\}$. Since $t \mapsto W_t(\omega)$ are continuous for all $\omega \in \Omega$, the same applies for $t \mapsto X_t(\omega)$. From Proposition [S.1.14] we thus deduce that $\eta = \tau_B$, the first hitting time of the closed set $B = [b, \infty)$ by the $\mathcal{F}_t^W$-adapted S.P. $\{X_t, t \geq 0\}$, is indeed an $\mathcal{F}_t^W$-stopping time, as claimed.

(b) Fixing $s > 0$ and $r \leq 0$ we note that $\theta = \theta(r, s) = \sqrt{r^2 + 2s} - r$ is merely the positive root of the quadratic equation $\frac{1}{4} \theta^2 + r \theta - s = 0$. Hence, by part (a) of Exercise [S.2.7]

$$Y_t = \exp(\theta X_t - st) = \exp(\theta W_t + (r \theta - s)t) = \exp(\theta W_t - \frac{\theta^2 t}{2})$$

is a non-negative $\mathcal{F}_t^W$-martingale of continuous sample functions. Consequently, by Corollary [S.2.29] the same applies for the stopped process $V_t = Y_{t \land \eta}$ (for we have seen in part (a) that $\eta$ is an $\mathcal{F}_t^W$-stopping time). Since $X_0 = 0 < b$ and $t \mapsto X_t$ is continuous, by the definition of $\eta$ necessarily $X_{t \land \eta} \leq b$ for all $t \geq 0$. Further,
Further, almost surely $\eta(\omega) < \infty$ then $X_t(\omega) = b$ and consequently $Y_\eta = \exp(\theta b - s\eta)$. With $\theta \geq 0$ it follows that $V_t < \exp(\theta X_{t\eta}) \leq e^{\theta b}$ is uniformly bounded, hence a U.I. martingale.

As such, we get from Doob’s optional sampling theorem that $Y_\eta := \limsup_{t \to \infty} V_t$ is integrable and $EY_\eta = EY_0 = 1$ (see part (b) of Exercise 8.2.30). Recall that $s > 0$, so if $\eta(\omega) = \infty$ then $V_t(\omega) \leq e^{\theta b-st} \to 0$ as $t \to \infty$. Thus, the identity $Y_\eta = \exp(\theta b - s\eta)$ extends to all $\omega \in \Omega$, from which we conclude that

$$1 = EY_\eta = E[e^{\theta b - s\eta}] = e^{\theta b}E[e^{-s\eta}].$$

That is, $E[e^{-s\tau_r^{b,c}(\omega)}] = e^{-\theta(r,s)b}$ for all $s, b > 0$ and $r \leq 0$.

(c). Since $\theta(r,s) \to \theta(r,0) = -2r$ and $e^{-s\eta} \uparrow I_{\eta<\infty}$ as $s \downarrow 0$, by monotone convergence we get from part (b) that $P(\tau_r^{b,c} < \infty) = e^{2rb}$.

(d). Setting $r = 0$ we have from part (c) that $P(\tau_b < \infty) = 1$ for all $b$, where $\tau_b = \inf\{t \geq 0 : W_t \geq b\}$. Consequently, the event $A = \bigcup_n \{\omega : \tau_n(\omega) < \infty\}$ occurs with probability one. Evidently, $A$ is merely the event $\limsup_{t \to \infty} W_t = \infty$, which thus happens almost surely. Considering this result in the case of the Brownian motion $\tilde{W}_t^{(1)} = -W_t$ (see (a) of Exercise 9.1.1), we further find that a.s. $\liminf_{t \to \infty} W_t = -\limsup_{t \to \infty} W_t^{(1)} = -\infty$.

**Solution.** By simplifying notations, fixing $r \in \mathbb{R}$ and $a, b > 0$ we hereafter drop the superscript $(r)$ subscripts $a, b$.

(a). By part (d) of Exercise 8.2.35 w.p.1. as $t \to \infty$ both $\limsup W_t = \infty$ and $\liminf W_t = -\infty$. If $r \leq 0$, then $\liminf Z_t \leq \liminf W_t = -\infty$, and if $r \geq 0$, then $\limsup Z_t \leq \limsup W_t = \infty$. So, in either case, almost surely $Z_t$ exits the interval $(-a, b)$ within a finite time $\tau = \tau_{a,b}^{(r)}$. By the same argument as in proof of part (a) of Exercise 8.2.35 $\tau$ is merely the first hitting time $\tau_B$ for $Z_t$ and the closed set $B = (-\infty, -a] \cup [b, \infty)$, hence an $\mathcal{F}_t^W$-stopping time.

From the definition of $\tau$ and continuity of $t \mapsto Z_t$ we deduce that $Z_{t \wedge \tau} \in [-a, b]$ is uniformly bounded. We thus have the uniformly bounded, hence U.I. stopped $\mathcal{F}_t^W$-martingale $Y_{t \wedge \tau}$ for $Y_t = u_0(t, W_t, -2r) = \exp(-2rZ_t)$ (as in part (b) of Exercise 8.2.35 except for taking now $s = 0$ and $\theta = -2r$). Consequently, by Doob’s optional sampling theorem $EY_\tau = EY_0$ (for example, see part (b) of Exercise 8.2.30). Further, if $\tau$ is finite (which occurs a.s.), then $Z_\tau \in [-a, b]$ and $Y_\tau \in \{e^{2ra}, -e^{2rb}\}$, respectively. So, setting $A = \{Z_\tau = -a\}, B = \{Z_\tau = b\}$ and $p_- = P(A)$ we have that

$$1 = EY_0 = EY_\tau = p_-e^{2ra} + (1 - p_-)e^{-2rb},$$

yielding the stated formula for $p_-$ in case $r \neq 0$. Now, if $r = 0$ then $Z_t = W_t$ is a martingale, so $Z_{t \wedge \tau}$ is a U.I. martingale and by the optional stopping theorem

$$0 = EZ_0 = EZ_\tau = -ap_- + b(1-p_-),$$

namely, $p_- = b/(a+b)$ when $r = 0$, as claimed.

(b). Here $r = 0$ and for $s \geq 0$ and $c = \pm 1$ we get upon applying Doob’s optional stopping theorem for the MG $Y_t = u_0(t, W_t, c\sqrt{2s}) = \exp(c\sqrt{2s}W_t - st)$ (such that $Y_{t \wedge \tau}$ is uniformly bounded), that

$$E[e^{c\sqrt{2s}W_{\tau - s\eta}}] = EY_\tau = EY_0 = 1.$$ 

Further, almost surely $I_A + I_B = 1$ and

$$e^{c\sqrt{2s}W_\tau} = e^{-ac\sqrt{2s}}I_A + e^{bc\sqrt{2s}}I_B.$$
resulting for \( c = \pm 1 \), with the identities
\[
e^{-ac\sqrt{2s}} E[e^{-st} I_A] + e^{bc\sqrt{2s}} E[e^{-st} I_B] = 1.
\]
We write these as a matrix equation
\[
\begin{bmatrix} e^{-a\sqrt{2s}} & e^{b\sqrt{2s}} \\ e^{a\sqrt{2s}} & e^{-b\sqrt{2s}} \end{bmatrix} \begin{bmatrix} E(e^{-st} I_A) \\ E(e^{-st} I_B) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
The determinant of this two-dimensional linear system is \( \Delta = -2 \sinh((a+b)\sqrt{2s}) \neq 0 \), so inverting the corresponding matrix we conclude that
\[
L_\tau(s) = E e^{-st} = \left[ 1, 1 \right] \frac{E(e^{-st} I_A)}{\begin{bmatrix} e^{-b\sqrt{2s}} \\ -e^{a\sqrt{2s}} \end{bmatrix} + \begin{bmatrix} e^{b\sqrt{2s}} \\ -e^{-a\sqrt{2s}} \end{bmatrix}} \frac{1}{\sinh((a+b)\sqrt{2s})}.
\]
(c). We know from part (b) that \( L_\tau(s) = C(s)/D(s) \), where
\[
C(s) = \frac{\sinh(a\sqrt{2s}) + \sinh(b\sqrt{2s})}{(a+b)\sqrt{2s}} = \sum_{k=0}^{\infty} c_k s^k,
\]
\[
D(s) = \frac{\sinh((a+b)\sqrt{2s})}{(a+b)\sqrt{2s}} = \sum_{k=0}^{\infty} d_k s^k,
\]
are analytic functions, with \( D(0) = C(0) = d_0 = c_0 = 1 \) and the positive coefficients \( c_k = 2^k (a^{2k+1} + b^{2k+1})/((a+b)(2k+1)! \) and \( d_k = 2^k (a+b)^{2k}/(2k+1)! \). Thus, \( L_\tau(s) \) is infinitely differentiable at \( s = 0 \) and admits a converging power-series expansion \( L_\tau(s) = M(s) \), where \( M(s) = \sum_{k=0}^{\infty} m_k s^k \) for \( m_0 = 1 \), some finite \( m_k, k \geq 1 \) and all \( s \in \mathbb{R}_+ \). Recall part (b) of Exercise 3.2.40 that \( E^k = (-1)^k k! m_k \), so your goal here is merely to verify that \( m_1 = -ab \) and \( m_2 = ab(a^2 + 3ab + b^2)/6 \). The shortest way to accomplish this task is to compare the coefficients of \( s \) and \( s^2 \) on both sides of the power-series identity \( C(s) = D(s) M(s) \). With \( d_0 = c_0 = m_0 = 1 \), doing so yields the linear equations \( c_1 = d_1 + m_1 \) and \( c_2 = m_1 d_1 + d_2 + m_2 \). Now, as \( d_1 = (a+b)^2/3 \) and \( c_1 = (a^2 - ab + b^2)/3 \), our first equation results with \( m_1 = c_1 - d_1 = -ab \). Then, as \( d_2 = (a+b)^4/30 \) and \( c_2 = (a^4 - a^3b + a^2b^2 - ab^3 + b^4)/30 \), our second equation results with
\[
m_2 = abd_1 + c_2 - d_2 = \frac{1}{3} ab(a+b)^2 \left( -\frac{1}{6} ab(a^2 + ab + b^2) = \frac{ab}{6} (a^2 + 3ab + b^2) \right),
\]
and so we are done (much faster than by the alternative of directly computing the first two derivatives of \( L_\tau(s) \) at \( s = 0 \).

**Solution.**

(a). Recall that \( \mathcal{F}_t^Y \equiv \sigma(Y_s : s \leq t) \) is generated by sets of form
\[
\{ Y_{t_1} \in \tilde{A}_1, \ldots, Y_{t_n} \in \tilde{A}_n \}, \quad \tilde{A}_i \in \tilde{S}, \quad 0 \leq t_1 < \ldots < t_n \leq t.
\]
Each mapping \( \Phi_\tau \) is invertible so such set can be re-expressed as
\[
\{ X_{u(t_1)} \in A_1, \ldots, X_{u(t_n)} \in A_n \},
\]
where \( A_i = \Phi_{\tau(t_i)}^{-1} \tilde{A}_i \in \mathcal{S} \) since \( \Phi_{t_i} \) is measurable. With \( \Phi_\tau \) having measurable inverse, for any \( A \in \mathcal{S} \) and \( s \geq 0 \) there exists \( \tilde{A}_s \in \tilde{S} \) such that \( A = \Phi_{s}^{-1}(\tilde{A}_s) \), so
the sets of the form \( \{ X_{u(t)} \in A_1, \ldots, X_{u(t_n)} \in A_n \} \) with \( A_i = \Phi_{t_i}^{-1}A_i \) and \( t_i \leq t \) generate \( \mathcal{F}_{u(t)}^X \). Consequently, \( \mathcal{F}_t^Y = \mathcal{F}_{X_{u(t)}}^X \).

Next, let \( p_{st} : \tilde{S} \times \tilde{S} \to [0, 1] \) denote the transition probabilities of the Markov process \( (X_t, \mathcal{F}_t^X, t \geq 0) \). Then, for any \( s < t \),

\[
P(Y_t \in \tilde{B} | \mathcal{F}_t^Y) = P(X_{u(t)} \in \Phi_{t_i}^{-1}\tilde{B} | \mathcal{F}_{u(t)}^X) = \tilde{p}_{st}(Y_s, \tilde{B})
\]

where

\[
\tilde{p}_{st} : \tilde{S} \times \tilde{S} \to [0, 1], \quad (y, \tilde{B}) \mapsto \tilde{p}_{u(s),u(t)}(\Phi_{s}^{-1}y, \Phi_{t}^{-1}\tilde{B}).
\]

It is easy to see that \( \tilde{p}_{st} \) is a valid transition probability. Indeed, for each \( y \in \tilde{S} \) fixed, \( \tilde{p}_{st}(y, \cdot) \) is a probability measure on \( \tilde{S} \) since it is the push-forward via \( \Phi_t \) of \( \tilde{p}_{u(s),u(t)}(\Phi_{s}^{-1}y, \cdot) \). Similarly, for each \( \tilde{B} \in \tilde{S} \), \( \tilde{p}_{st}(\cdot, \tilde{B}) \) is \( \tilde{S} \)-measurable since it is the composition of the measurable mappings \( \Phi_{s}^{-1} : \tilde{S} \to S \) and \( p_{u(s),u(t)}(\cdot, \Phi_{t}^{-1}\tilde{B}) : S \to [0, 1] \). Finally, we verify the Chapman-Kolmogorov equations:

\[
\tilde{p}_{st}(y, \tilde{B}) = \int_{\tilde{S}} \tilde{p}_{st}(y', \tilde{B}) \tilde{p}_{t_1 t_2}(y, dy')
\]

\[
= \int_{\tilde{S}} p_{u(t_2)u(t_3)}(x, \Phi_{t_3}^{-1}\tilde{B}) p_{u(t_3)u(t_2)}(\Phi_{t_2}^{-1}y, dx)
\]

\[
= p_{u(t_1)u(t_2)} p_{u(t_2)u(t_3)}(\Phi_{t_3}^{-1}y, \Phi_{t_2}^{-1}\tilde{B})
\]

\[
= p_{u(t_1)u(t_2)}(\Phi_{t_1}^{-1}y, \Phi_{t_2}^{-1}\tilde{B}) = \tilde{p}_{t_1 t_2}(y, \tilde{B}).
\]

This proves that \( (Y_t, \mathcal{F}_t^Y)_{t \geq 0} \) is a Markov process on state space \((\tilde{S}, \tilde{S})\) with transition probabilities \( \tilde{p}_{st} \).

(b). By part (a) with \( \Phi_t \equiv \Phi_0 \), \( u(t) \equiv t \) and \( p_{st}(\cdot, \cdot) = p_{t-s}(\cdot, \cdot) \) we have that \( (Z_t, \mathcal{F}_t^Z)_{t \geq 0} \) is a Markov process of transition probabilities

\[
q_{st}(z, \tilde{B}) = p_{st}(\Phi_0^{-1}z, \Phi_0^{-1}\tilde{B}) = p_{t-s}(\Phi_0^{-1}z, \Phi_0^{-1}\tilde{B})
\]

hence \( q_{st}(\cdot, \cdot) = q_{t-s}(\cdot, \cdot) \) and \( (Z_t, \mathcal{F}_t^Z) \) is also a homogeneous Markov process.

Additional solutions provided.

**Solution.** **S.2.32**

(a). Recall part (b) of Exercise **S.1.10** that \( \theta_u := u + \tau \) are bounded \( \mathcal{F}_t \)-stopping times. Hence,

\[
\mathcal{G}_u = \mathcal{F}_{\theta_u} = \{ A \in \mathcal{F}_\infty : A \cap \{ u + \tau \leq t \} \in \mathcal{F}_t, \ \forall t \geq 0 \}
\]

\[
= \{ A \in \mathcal{F}_\infty : A \cap \{ \tau \leq s \} \in \mathcal{F}_{s+u}, \ \forall s \geq 0 \}.
\]

Clearly, the \( \sigma \)-algebras \( \mathcal{G}_u \) are non-decreasing in \( u \), hence \( \{ \mathcal{G}_t, t \geq 0 \} \) is a filtration. Further, by definition

\[
\mathcal{G}_{u+} = \bigcup_{\varepsilon > 0} \mathcal{G}_{u+\varepsilon} = \{ A \in \mathcal{F}_\infty : A \cap \{ \tau \leq s \} \in \mathcal{F}_{s+u+\varepsilon}, \ \forall s \geq 0, \ \varepsilon > 0 \}
\]

Comparing these two identities note that our assumption that \( \mathcal{F}_{s+u} = \mathcal{F}_{(s+u)^+} \) for all \( s, u \geq 0 \), yields that \( \mathcal{G}_u = \mathcal{G}_{u+} \) for all \( u \geq 0 \), as claimed.

(b). Since \( \tau \leq c \) for some non-random \( c < \infty \) we know from Proposition **S.1.13** that \( X_{\theta_\varepsilon} \in m\mathcal{F}_\varepsilon = m\mathcal{G}_\varepsilon \) for all \( t \geq 0 \), so \( \{ Y_t = X_{\theta_u} - X_{\theta_u}, t \geq 0 \} \) is \( \mathcal{G}_t \)-adapted. Further, \( Z_s = |X_s| \) is a right-continuous \( \mathcal{F}_s \)-sub-MG (see Exercise **S.2.9**), and \( X_{u+\theta_u} = X_{\theta_u} \) for the non-random \( u = c + t \) (since \( \theta_t = \tau + t \leq u \)). Hence, by Doob’s optional
stopping theorem $\mathbf{E}[Z_u|\mathcal{F}_0] \geq Z_{\theta_b}$ (see part (a) of Exercise 8.2.30), and taking the expectation of both sides we deduce that $\mathbf{E}[X_{\theta_b}] \leq \mathbf{E}[X_u]$ is finite for any $t \geq 0$. That is, $\{\mathcal{Y}_t\}$ is also an integrable S.P. of right-continuous sample functions.

Fixing $t \geq s \geq 0$, upon applying the same reasoning for the right-continuous sub-MG $(X_v, \mathcal{F}_v)$ and the $\mathcal{F}_v$-stopping times $\theta_s \leq \theta_t$ (which are both bounded by $u$), we find that $\mathbf{E}[X_{\theta_t}|\mathcal{F}_{\theta_s}] \geq X_{\theta_s}$. Consequently, for any $t \geq s \geq 0$,

$$
\mathbf{E}[Y_t|\mathcal{G}_s] = \mathbf{E}[X_{\theta_t} - X_{\theta_s}|\mathcal{F}_{\theta_s}] = \mathbf{E}[X_{\theta_t}|\mathcal{F}_{\theta_s}] - X_{\theta_s} \geq X_{\theta_t} - X_{\theta_s} = Y_s,
$$

hence $(Y_t, \mathcal{G}_t)$ is a right-continuous sub-MG, as claimed.

**Solution. 8.2.38**

(a). With $W_i(t), i = 1, \ldots, k$, independent standard Brownian motions,

$$
M_t = R_t^2 - kt = \sum_{i=1}^{k} (W_i(t)^2 - t)
$$

is clearly integrable, of continuous sample functions and adapted to the filtration $\mathcal{F}_t^W = \sigma(\mathcal{F}_t^W, i = 1, \ldots, k)$. By Proposition 4.2.3 and the mutual independence of $\mathcal{F}_i^W, i = 1, \ldots, k$, each of the martingales $(W_i(t)^2 - t, t \geq 0)$ is also an $\mathcal{F}_t^W$-martingale. Consequently, so is their sum $M_t$, as for any $0 \leq s \leq t$ by linearity of the C.E.

$$
\mathbf{E}[M_t|\mathcal{F}_s^W] = \sum_{i=1}^{k} \mathbf{E}[W_i(t)^2 - t|\mathcal{F}_s^W] = \sum_{i=1}^{k} (W_i(s)^2 - s) = M_s.
$$

Next, from Proposition 8.1.15 we know that $\theta_{b_0}$, being the first hitting time of the closed set $B = [b, \infty)$ by the $\mathcal{F}_t^W$-adapted process $R_t$ of continuous sample functions, is an $\mathcal{F}_t^W$-stopping time. Further, since $R_t \geq |W_1(t)|$, clearly $\theta_b \leq \tau_{b,b}$ is a.s. finite (in view of part (a) of Exercise 8.2.30).

(b). Applying part (a) of Exercise 8.2.30 to the continuous $\mathcal{F}_t^W$-MG $M_t$ and stopping times $\theta_b \geq 0$, we deduce that $\mathbf{E}[M_{u \land \theta_b}|\mathcal{F}_0^W] = M_0 = 0$ for any non-random $u \geq 0$. Since $M_t = R_t^2 - kt$ and $\mathcal{F}_0^W = \{\Omega, \emptyset\}$, by linearity of the expectation this amounts to

$$
\mathbf{E}[R_{u \land \theta_b}^2] = k \mathbf{E}[u \land \theta_b]
$$

holding for all $u \geq 0$. By continuity of $t \mapsto R_t$ and the definition of $\theta_b$ we know that $u \mapsto R_{u \land \theta_b}$ is uniformly bounded by $b^2$, hence U.I. and if $\theta_b$ is finite (which by part (a) occurs w.p.1.), then it converges to $R_{\theta_b}^2$ as $u \to \infty$. Further, $u \land \theta_b \uparrow \theta_b$ as $u \uparrow \infty$, so by U.I. and monotone convergence,

$$
\mathbf{E}[R_{\theta_b}^2] = \lim_{u \to \infty} \mathbf{E}[R_{u \land \theta_b}^2] = k \lim_{u \to \infty} \mathbf{E}[u \land \theta_b] = k \mathbf{E}[\theta_b],
$$

and upon noting that $R_{\theta_b} = b$ (by continuity of $t \mapsto R_t$), we are done.