Hex, Random Turn Games, and the Infinity Laplacian

Abstract. The infinity Laplacian (informally, the “second derivative in the gradient direction”) is a simple yet mysterious operator with many applications, in particular to optimal Lipschitz extensions. Classical analysis of this operator is hampered by nonsmoothness of solutions. "Tug of war" is a two player random turn game played as follows: Given disjoint target sets $T_1$ and $T_2$ in the plane, and a token at $x$, toss a fair coin; the player who wins the coin toss moves the particle up to distance $r$ in the direction of his/her choice. This is repeated until the token reaches a target set $T_i$; player $i$ is then declared the winner. Write $u_r(x)$ for the probability that player 1 wins when both players play optimally. We show that as $r$ tends to 0, the functions $u_r(x)$ converge to the infinity harmonic function with boundary conditions 1 on $T_1$ and 0 on $T_2$. Our analysis of tug of war leads yields new estimates, and significant generalizations of several classical results about infinity Laplacians. I will also describe our original motivation for studying random-turn games: A variant of the game of Hex with a conformally-invariant limit.

(Talk based on joint work with Oded Schramm, Scott Sheffield and David Wilson.)