Brownian Motion on Disconnected Sets, Basic Hypergeometric Functions, and some Continued Fractions of Ramanujan

Abstract. A fundamental theorem of Paul Lévy characterises standard Brownian motion as the unique continuous martingale with the identity function as its quadratic variation process. It turns out that on any closed subset of the real line that is unbounded above and below there is a unique process that is a martingale, has the identity function as its quadratic variation process, and is “continuous” in the sense that its sample paths don’t skip over points. This process is automatically a Feller-Dynkin Markov process and its generator is a natural generalisation of the operator $f \mapsto \frac{1}{2}f''$.

An interesting special case occurs when the state space is the self-similar set $\{\pm q^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $q > 1$. The analysis of this process leads to continued fractions that appear in Ramanujan’s “lost” notebook and can be evaluated in terms of $q$-analogues of classical hypergeometric functions. Moreover, the distributions of various features of the process involve $q$-analogues of classical distributions such as the Poisson distribution that have appeared elsewhere in the literature.

This is joint work with Shankar Bhamidi, Ron Peled and Peter Ralph.