Large Deviations for Diffusions Interacting Through Their Ranks

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Abstract

We prove a large deviations principle (LDP) for systems of diffusions (particles) interacting through their ranks when the number of particles tends to infinity. We show that the limiting particle density is given by the unique solution of the appropriate McKean-Vlasov equation and that the corresponding cumulative distribution function evolves according to a nondegenerate generalized porous medium equation with convection. The large deviations rate function is provided in explicit form. This is the first instance of an LDP for interacting diffusions where the interaction occurs both through the drift and the diffusion coefficients and where the rate function can be given explicitly. In the course of the proof, we obtain new regularity results for tilted versions of such a generalized porous medium equation. © 2016 Wiley Periodicals, Inc.

1 Introduction

Systems of diffusion processes (particles) interacting through their ranks have recently received much attention. For a fixed number of particles $N \in \mathbb{N}$, these are given by the unique weak solution of the stochastic differential system (SDS),

$$\begin{aligned} \mathrm{d}X_i(t) &= \sum_{j=1}^{N} b_j \mathbb{1}_{\{X_i(t)=X_{(j)}(t)\}}\mathrm{d}t \\ &\quad + \sum_{j=1}^{N} \sigma_j \mathbb{1}_{\{X_i(t)=X_{(j)}(t)\}}\mathrm{d}W_i(t), \quad i = 1, \dots, N, \end{aligned} \tag{1.1}$$

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where $b_1, b_2, \dots, b_N$ are arbitrary real constants, $\sigma_1, \sigma_2, \dots, \sigma_N$ are arbitrary positive constants, $W_1, W_2, \dots, W_N$ are independent standard Brownian motions, and $X_{(1)}(t) \le X_{(2)}(t) \le \cdots \le X_{(N)}(t)$ are the ordered particles at time t. In this paper we study the behavior of the solution to (1.1) as N becomes large in the regime when $|b_{j+1} - b_j| + |\sigma_{j+1}^2 - \sigma_j^2| = O(N^{-1})$ for all j (see Assumption 1.2 below for the details). In other words, the drift and the diffusion coefficients of a particle change slowly as it changes its rank in the particle system.

The existence and uniqueness of the weak solution to (1.1) were shown in the work [3], which was motivated by questions in filtering theory. The system (1.1) has also reappeared in the context of stochastic portfolio theory under the name *first-order market model* (see the book [14] and the survey article [23]). In the latter context, our choice of the regime $|b_{j+1} - b_j| + |\sigma_{j+1}^2 - \sigma_j^2| = O(N^{-1})$ agrees with the economic intuition that a small change of a company's rank cannot lead to a large jump in the growth rate and the volatility coefficient of its market capitalization. Due to its central role in the analysis of capital distributions in financial markets and long-term portfolio performance therein, as well as its intriguing mathematical features, the ergodicity and sample path properties of this model have undergone a detailed analysis in the case where the number of particles is fixed (see [6, 7, 18–20]). Moreover, concentration properties of the solution to (1.1) for large values of N have been studied in [34], and an analogous infinite particle system has been constructed and analyzed in [33].

In [36] it was observed that the SDS (1.1) can be rewritten as

$$
\begin{aligned}
\mathrm{d}X_i(t) &= b\big(F_{\rho^N(t)}(X_i(t))\big)\mathrm{d}t \\
&\quad + \sigma\big(F_{\rho^N(t)}(X_i(t))\big)\mathrm{d}W_i(t), \quad i = 1, 2, \dots, N,
\end{aligned} \tag{1.2}
$$

where $\rho^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}$ is the $F_{\rho^N(t)}$ is the corresponding cumulative distribution function (CDF), and $b : [0,1] \to \mathbb{R}$, $\sigma : [0,1] \to (0,\infty)$ are functions satisfying $b(j/N) = b_j$, $\sigma(j/N) = \sigma_j$ for all $j = 1, 2, \dots, N$. The representation gives rise to questions on the large-N behavior of the empirical measure in (1.2), where the mathematical challenge is due to the discontinuity of the diffusion coefficients in (1.2). A law of large numbers (LLN) for $\rho^N(t)$ is obtained in [36] for nondecreasing $i \mapsto X_i(0)$, with $\{X_i(0) - X_1(0)\}$ chosen according to the stationary distribution of the process of spacings between consecutively ordered particles in (1.2) (in particular, assuming that b and σ are such that a stationary distribution exists). In this case, it was shown that the limiting particle measure path $t \mapsto \gamma(t)(\cdot)$ satisfies the *McKean-Vlasov equation*

$$
\begin{aligned}
\int_{\mathbb{R}} f\,\mathrm{d}\gamma(t) - \int_{\mathbb{R}} f\,\mathrm{d}\gamma(0) &= \\
&\int_0^t \mathrm{d}s \int_{\mathbb{R}} \Big[b(F_{\gamma(s)}(\cdot))f' + \frac{1}{2}\sigma(F_{\gamma(s)}(\cdot))^2 f''\Big]\mathrm{d}\gamma(s)
\end{aligned} \tag{1.3}
$$

for all Schwartz functions $x \mapsto f(x)$ and any $t \geq 0$ (hereafter $\mathrm{d}\gamma(s)$ is shorthand for integration with respect to the probability measure $\gamma(s)(\cdot)$ on $\mathbb{R}$ at the fixed time s). Further, the corresponding CDF-path $R^{(\gamma)}(t,x) := F_{\gamma(t)}(x)$ evolves according to the *nondegenerate generalized porous medium equation with convection* (see [37] and the references therein for a thorough treatment),

$$R_t = (\Sigma(R))_{xx} - (\Gamma(R))_x, \tag{1.4}$$

where $\Sigma(r) = \int_0^r \frac{1}{2}\sigma^2(u)\mathrm{d}u$ and $\Gamma(r) = \int_0^r b(u)\mathrm{d}u$.

In this paper, we establish a large deviations principle (LDP) for the sequence $\{\rho^N : N \in \mathbb{N}\}$ of paths $\{\rho^N(t) : t \in [0,T]\}$ of empirical measures, where $T > 0$ is arbitrary but fixed throughout. Among other things, this LDP implies the LLN for $\{\rho^N : N \in \mathbb{N}\}$. Such LLN and the LDP upper bound are shown under the mild regularity Assumption 1.2 on the functions b, σ and the initial empirical measures $\{\rho^N(0) : N \in \mathbb{N}\}$ (dispensing of the stationarity assumption on the process of spacings, which plays a crucial role in [36]). Our next definition is useful for stating Assumption 1.2.

DEFINITION 1.1. Let $M_1(\mathbb{R})$ denote the space of Borel probability measures on $\mathbb{R}$, endowed with the Lévy distance metric

$$\begin{aligned} d_L(\alpha_1,\alpha_2) := \inf\{\epsilon > 0 \mid \alpha_1(O) \leq \alpha_2(O_\epsilon) + \epsilon, \\ \alpha_2(O) \leq \alpha_1(O_\epsilon) + \epsilon \ \ \forall \text{ open } O \subset \mathbb{R}\} \end{aligned}$$

(where O_ϵ stands for the ϵ-neighborhood of O in $\mathbb{R}$). Then, for $\iota \in [0,1]$ let $M_1^{(\iota)}(\mathbb{R})$ denote the subset of all $\mu \in M_1(\mathbb{R})$ such that $\int_{\mathbb{R}} |x|^{1+\iota}\mathrm{d}\mu < \infty$ and $\frac{\mathrm{d}\mu}{\mathrm{d}x} \in L^{q_0}(\mathbb{R})$ for some $q_0 > 1$.

Assumption 1.2. $\sigma_j = \sigma(j/N)$ and $b_j = b(j/N)$, where:

(a) The function $A := \frac{1}{2}\sigma^2$ is uniformly bounded below by some $\underline{a} > 0$, and A' is bounded on $[0,1]$.
(b) The function $b(\cdot)$ is Lipschitz-continuous on $[0,1]$.
(c) As $N \to \infty$ the deterministic initial empirical measures $\{\rho^N(0)\}$ converge weakly to some $\rho_0 \in M_1^{(\iota_\star)}(\mathbb{R})$, $\iota_\star \in (0,1]$, and

$$\sup_{N\in\mathbb{N}} \int_{\mathbb{R}} |x|^{1+\iota_\star}\,\mathrm{d}\rho^N(0) < \infty.$$

(d) The function A' is Lipschitz-continuous on $[0,1]$.

Throughout the paper we let $\mathscr{C} = C([0,T], M_1(\mathbb{R}))$ stand for the space of continuous functions from $[0,T]$ to $M_1(\mathbb{R})$ endowed with the metric

$$d(\gamma_1(\cdot),\gamma_2(\cdot)) = \sup_{t\in[0,T]} d_L(\gamma_1(t),\gamma_2(t)). \tag{1.5}$$

Further, throughout we adopt the convention that $\frac{0}{0} = 0$, and identify each $\gamma \in \mathscr{C}$ with the corresponding CDF-path $R^{(\gamma)}(t,x) := F_{\gamma(t)}(x)$. The following spaces of functions then play a major role in our LDP on $\mathscr{C}$.

DEFINITION 1.3. Let $\widetilde{\mathscr{S}}$ denote the space of functions g on $\mathbb{R}_T := [0,T] \times \mathbb{R}$ that are infinitely differentiable and such that, for all $t \in [0,T]$, $g(t,\cdot)$ is a Schwartz function on $\mathbb{R}$. Next, let

$$\widetilde{\mathscr{A}} := \{\gamma \in \mathscr{C} : R^{(\gamma)} \in C_b(\mathbb{R}_T),\ \gamma(0) \in M_1^{(0)}(\mathbb{R}),\\ t \mapsto \textstyle\int_{\mathbb{R}} g(t,\cdot)\mathrm{d}\gamma(t) \text{ abs. cont. on } [0,T]\ \forall g \in \widetilde{\mathscr{S}}\},$$

and for each $\iota \in (0,1]$, $\mu \in M_1^{(0)}(\mathbb{R})$, define its subset

$$\mathscr{A}_{\iota,\mu} := \{\gamma \in \widetilde{\mathscr{A}} : \gamma(0) = \mu,\ \textstyle\int_{\mathbb{R}_T} |x|^{1+\iota}\, \mathrm{d}\gamma(t)\mathrm{d}t < \infty\}. \tag{1.6}$$

Further, we use

$$\mathscr{F}_q := \Big\{R = R^{(\gamma)} : \gamma \in \mathscr{C},\ R_t, R_{xx} \in L^q(\mathbb{R}_T),\ R_x \in L^3(\mathbb{R}_T),\\ \frac{R_t^2}{R_x}, \frac{R_{xx}^2}{R_x} \in L^1(\mathbb{R}_T)\Big\} \tag{1.7}$$

with $\mathscr{F} := \mathscr{F}_{3/2}$ and

$$\widetilde{J}(\gamma) := \begin{cases} \frac{1}{4}\left\| \frac{R_t - (A(R)R_x)_x + b(R)R_x}{(A(R)R_x)^{1/2}} \right\|^2_{L^2(\mathbb{R}_T)} & \text{if } \gamma \in \widetilde{\mathscr{A}},\ R = R^{(\gamma)} \in \mathscr{F}, \\ \infty & \text{otherwise,} \end{cases} \tag{1.8}$$

with $J_{\iota,\mu}(\gamma)$ defined as in (1.8) except for replacing there $\widetilde{\mathscr{A}}$ by the smaller $\mathscr{A}_{\iota,\mu}$.

Our main result then reads as follows:

THEOREM 1.4. *Under Assumption* 1.2 *with* $\iota_\star = 1$, *the sequence* $\{\rho^N : N \in \mathbb{N}\}$ *satisfies the LDP on* $\mathscr{C}$ *with scale* N *and the good rate function* $J_{1,\rho_0}(\cdot)$ *of Definition* 1.3.

Remark 1.5. As shown in Proposition 2.6 (and Corollary 2.3), the exponential tightness of $\{\rho^N\}$ and the LDP upper bound of Theorem 1.4 with rate $J_{\iota_\star,\rho_0}(\cdot)$ apply for any value of $\iota_\star > 0$ in Assumption 1.2(c) and do not require part (d) of Assumption 1.2.

In view of the preceding remark we have the following LLN.

COROLLARY 1.6. *Under Assumption* 1.2(a)–(c) *for some* $\iota_\star \in (0,1]$, *the sequence* $\{\rho^N : N \in \mathbb{N}\}$ *converges almost surely to the unique path* $\gamma_\star \in \mathscr{A}_{\iota_\star,\rho_0}$ *for which* $R^{(\gamma_\star)} \in \mathscr{F}$ *is a generalized solution of the Cauchy problem*

$$R_t = (A(R)R_x)_x - b(R)R_x,\ R(0,\cdot) = F_{\rho_0}(\cdot). \tag{1.9}$$

PROOF. Setting $J := J_{\iota_\star,\rho_0}$, recall Remark 1.5 that the exponentially tight $\{\rho^N\}$ satisfy the LDP upper bound in the metric space $(\mathscr{C}, d)$ with some rate function $I_{\iota_\star,\rho_0}(\cdot) \geq J(\cdot)$. Necessarily, $I_{\iota_\star,\rho_0}$ has compact, nonempty, level sets and in particular $J^{-1}(0) = \{\gamma \in \mathscr{C} : J(\gamma) = 0\}$ is nonempty (and precompact). Considering the LDP upper bound for the complement of any finite δ-cover of $J^{-1}(0)$, we further deduce by the first Borel-Cantelli lemma that a.s. $d(\rho^N, J^{-1}(0)) \to 0$. From the explicit formula (1.8) we know that $J^{-1}(0) \subseteq \mathscr{A}_{\iota_\star,\rho_0}$, and furthermore, to each $\gamma \in J^{-1}(0)$ corresponds $R^{(\gamma)} \in \mathscr{F}$, which is a nonnegative continuous bounded generalized solution of the problem (1.9) in the sense of [17, def. 4]. Recall [17, theorem 4] that such a generalized solution is unique. Consequently, $J^{-1}(0) = \{\gamma_\star\}$ is a single point to which ρ^N converges a.s. when $N \to \infty$. □

In [8], the authors prove an LDP for systems of diffusions with the same constant diffusion coefficient, where each drift coefficient is a continuous function of the value of the diffusion and the empirical measure of the whole system. In this context the (local) LDP is established by a clever application of Girsanov's theorem, which allows one to move from the system of interacting diffusions to the corresponding system of independent diffusions (in the event that the path of empirical measures is near a deterministic path of probability measures). Such an approach is not viable in our case because of the interaction through the diffusion coefficients in (1.2). Moreover, the discontinuity of the drift and the diffusion coefficients presents an additional challenge. Even on the level of the LLN as in Corollary 1.6, previous works had to assume that there is no interaction through the diffusions coefficients (see [4, 5, 21] and the references therein), or be restricted to special initial conditions (see [36]). We overcome these challenges but remark that our analysis relies on the particular form of the drift and the diffusion coefficients in (1.2).

A crucial part of the proof of Theorem 1.4 is devoted to the study of generalized solutions to porous medium equations with tilt

$$R_t = (A(R)R_x)_x + h\,A(R)R_x. \tag{1.10}$$

The following regularity result, which we need in the proof of Theorem 1.4, is also of independent interest.

THEOREM 1.7. *Let $R \in C_b(\mathbb{R}_T)$ be such that, for every $t \in [0, T]$, the function $R(t, \cdot)$ is the CDF of a probability measure $\gamma(t)$. Suppose that R is a generalized solution to* (1.10) *with initial condition $R(0, \cdot) = F_\mu(\cdot)$, where $A(\cdot)$ satisfies Assumption* 1.2(a)*, $\mu \in M_1^{(0)}(\mathbb{R})$, and h is a function on $\mathbb{R}_T$ such that*

$$\int_{\mathbb{R}_T} h^2(t, x)\mathrm{d}\gamma(t)\mathrm{d}t < \infty. \tag{1.11}$$

If, in addition, $\gamma(\cdot)$ satisfies the moment condition (1.6) *for some $\iota > 0$, then $\gamma \in \mathscr{A}_{\iota,\mu}$ and $R \in \mathscr{F}_q$ of* (1.7) *for all $\frac{6}{5} \leq q \leq \frac{3}{2}$.*

2 Outline: Proofs of Theorems 1.4 and 1.7

In this section we establish Theorems 1.4 and 1.7 as consequences of Propositions 2.2 and 2.5–2.7. The latter are in turn proved in the following five sections, in the order in which they are stated here. We note in passing that the proofs of Propositions 2.2 and 2.5 are of analytic nature (relying for proving Proposition 2.2 on results from [25, 26, 29] about parabolic equations with non-smooth coefficients), whereas those of Propositions 2.6 and 2.7 are mostly probabilistic, involving tools from large deviations theory and stochastic analysis. More precisely, the local large deviations upper bound of Proposition 2.6 is established by integrating suitable test functions against ρ^N, proving a Freidlin-Wentzell-type local large deviations upper bound for the resulting processes and optimizing over such test functions; the local large deviations lower bound of Proposition 2.7 is shown via a tilting argument that relies on an appropriate Girsanov change of measure.

We proceed with a few notations and definitions that are used throughout this paper. First, we write m for the Lebesgue measure on $\mathbb{R}_T = [0, T] \times \mathbb{R}$, (α, f) for $\int_{\mathbb{R}} f \,\mathrm{d}\alpha$, any f in the space of continuous bounded functions $C_b(\mathbb{R})$ and any $\alpha \in M_1(\mathbb{R})$, with $(\alpha, f)(s) = \int_{\mathbb{R}} f(s, \cdot)\mathrm{d}\alpha(s, \cdot)$, in case of $s \mapsto f(s, \cdot) \in C_b(\mathbb{R})$ and $s \mapsto \alpha(s, \cdot) \in M_1(\mathbb{R})$ (or more generally, whenever $f(s, \cdot)$ is integrable with respect to $\alpha(s, \cdot)$). We further let $\widetilde{\mathscr{S}}_x = \{g_x : g \in \widetilde{\mathscr{S}}\}$ denote the space of spatial derivatives of test functions from $\widetilde{\mathscr{S}}$.

DEFINITION 2.1. Setting

$$\mathscr{R}^\gamma g = g_t + b(R^{(\gamma)})g_x + A(R^{(\gamma)})g_{xx}, \tag{2.1}$$

with the functional

$$\Phi_\gamma(t, g) = (\gamma, g)(t) - (\gamma, g)(0) - \int_0^t (\gamma, \mathscr{R}^\gamma g)(s)\mathrm{d}s \tag{2.2}$$

on $\widetilde{\mathscr{S}}$, $\Phi_\gamma(g) := \Phi_\gamma(T, g)$, and inner product

$$(f, g)_\gamma = \int_{\mathbb{R}_T} f_x \, g_x \, A(R^{(\gamma)})\mathrm{d}\gamma(t)\mathrm{d}t \tag{2.3}$$

on $\widetilde{\mathscr{S}}$, we consider the (rate) functions on $\mathscr{C}$ given by

$$I_{\iota,\mu}(\gamma) = \begin{cases} \sup_{g \in \widetilde{\mathscr{S}}}[\Phi_\gamma(g) - (g, g)_\gamma], & \gamma \in \mathscr{A}_{\iota,\mu}, \\ \infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Theorem 1.7 is a direct consequence of the following proposition (which is also key in proving Theorem 1.4).

PROPOSITION 2.2. *Suppose Assumption* 1.2(a) *holds and the function* $b(\cdot)$ *is uniformly bounded. If* $I_{\iota,\mu}(\gamma) < \infty$ *for some* $\iota > 0$ *and* $\mu \in M_1^{(0)}(\mathbb{R})$*, then* $R^{(\gamma)} \in \mathscr{F}_q$ *for all* $\frac{6}{5} \leq q \leq \frac{3}{2}$*. Namely,* $R^{(\gamma)} = R$ *such that*

(A) $R_x \in L^3(\mathbb{R}_T)$,
(B) $R_t, R_{xx} \in L^q(\mathbb{R}_T)$ *for all* $\frac{6}{5} \le q \le \frac{3}{2}$,
(C) $\int_{\mathbb{R}_T} \frac{R_{xx}^2}{R_x}\, \mathrm{d}m < \infty$, $\int_{\mathbb{R}_T} \frac{R_t^2}{R_x}\, \mathrm{d}m < \infty$.

PROOF OF THEOREM 1.7. After integration by parts in space, we see that having $R = R^{(\gamma)}$ as a generalized solution of (1.10) is equivalent to

$$\Phi_\gamma(t, g) = -\int_0^t (\gamma, b(R)g_x + hA(R)g_x)(s)\mathrm{d}s \tag{2.5}$$

for any $g \in \widetilde{\mathscr{S}}$ and $t \in [0, T]$. In particular, upon comparing (2.2) and (2.5), we deduce that $t \mapsto (\gamma, g)(t)$ is absolutely continuous for any $g \in \widetilde{\mathscr{S}}$, and thus $\gamma \in \mathscr{A}_{\iota,\mu}$ (for $R \in C_b(\mathbb{R}_T)$, and the moment and initial conditions have all been assumed in Theorem 1.7). Further, taking here $b \equiv 0$ without loss of generality, we see that for γ as in Theorem 1.7,

$$\begin{aligned} I_{\iota,\mu}(\gamma) &= \sup_{f \in \widetilde{\mathscr{S}}_x} \left[\int_{\mathbb{R}_T} (-hA(R)f - A(R)f^2)\mathrm{d}\gamma(t)\mathrm{d}t \right] \\ &\le \sup_{f \in L^2(\mathbb{R}_T, \mathrm{d}\gamma(t)\mathrm{d}t)} \left[\int_{\mathbb{R}_T} (-hA(R)f - A(R)f^2)\mathrm{d}\gamma(t)\mathrm{d}t \right]. \end{aligned}$$

The latter supremum is attained for $f = -\frac{1}{2}h$ and its value is finite due to our assumption (1.11). Consequently, in this case $I_{\iota,\mu}(\gamma) < \infty$ and by Proposition 2.2 such $R = R^{(\gamma)}$ satisfies the regularity properties (1.7) for all $\frac{6}{5} \le q \le \frac{3}{2}$, as claimed. □

We start the proof of Theorem 1.4 by establishing the following corollary of Proposition 2.2.

COROLLARY 2.3. *Suppose $A(\cdot)$ satisfies Assumption* 1.2(a), $\mu \in M_1^{(0)}(\mathbb{R})$, *and $b(\cdot)$ is uniformly bounded. If $I_{\iota,\mu}(\gamma) < \infty$, then $J_{\iota,\mu}(\gamma) = I_{\iota,\mu}(\gamma)$. In particular, $J_{\iota,\mu}(\gamma) \le I_{\iota,\mu}(\gamma)$ for all $\gamma \in \mathscr{C}$.*

PROOF. Fixing $\gamma \in \mathscr{A}_{\iota,\mu}$ with

$$I_{\iota,\mu}(\gamma) = \sup_{g \in \widetilde{\mathscr{S}}} [\Phi_\gamma(g) - (g, g)_\gamma] < \infty,$$

consider the Hilbert space $\mathbb{H}$ given by identifying and completing $\widetilde{\mathscr{S}}$ under the seminorm corresponding to the inner product $(\cdot, \cdot)_\gamma$ of (2.3). By scaling, the linear functional $\Phi_\gamma(\cdot)$ is bounded on $\widetilde{\mathscr{S}}$ by $2\sqrt{I_{\iota,\mu}(\gamma)}$ times this seminorm, and with $A(\cdot)$ uniformly bounded below, if $(g, g)_\gamma = 0$ then by (2.1)–(2.3) also $\Phi_\gamma(g) = 0$. Hence, there exists a unique bounded linear functional $\overline{\Phi}_\gamma$ on $\mathbb{H}$ which coincides with Φ_γ on $\widetilde{\mathscr{S}}$. Now, by the Riesz representation theorem, there is a unique element $\tilde{h} \in \mathbb{H}$, which satisfies $\overline{\Phi}_\gamma(g) = (\tilde{h}, g)_\gamma$ for all $g \in \mathbb{H}$. Combining this with the fact that $\widetilde{\mathscr{S}}$ is by definition dense in $\mathbb{H}$, we obtain that

$$I_{\iota,\mu}(\gamma) = \sup_{g\in\mathbb{H}} \big[\overline{\Phi}_\gamma(g) - (g,g)_\gamma\big] = \sup_{g\in\mathbb{H}} \big[(\widetilde{h},g)_\gamma - (g,g)_\gamma\big] = \frac{1}{4}(\widetilde{h},\widetilde{h})_\gamma. \tag{2.6}$$

Furthermore, by the definition of $\widetilde{h}$ and Φ_γ, we have that $\widetilde{h}_x \in L^2(\mathbb{R}_T, \mathrm{d}\gamma(t)\mathrm{d}t)$ satisfies

$$\Phi_\gamma(t,g) = \int_0^t (\gamma, A(R^{(\gamma)})\widetilde{h}_x g_x)(s)\mathrm{d}s \tag{2.7}$$

for $t = T$ and any $g \in \widetilde{\mathscr{S}}$. In particular, considering Schwartz functions g supported on $\mathbb{R}_t$ we have that (2.7) also applies for any $t \in [0,T]$. Comparing this with (2.5) we deduce that $R = R^{(\gamma)}$ is a generalized solution of the PDE (1.10) for

$$h = -\widetilde{h}_x - \frac{b(R)}{A(R)}. \tag{2.8}$$

By the assumed boundedness of $\frac{b(\cdot)}{A(\cdot)}$, clearly $h \in L^2(\mathbb{R}_T, \mathrm{d}\gamma(t)\mathrm{d}t)$. By Theorem 1.7, this implies in turn that R_t, R_{xx}, and the $L^1(\mathbb{R}_T)$ density R_x are elements of $L^{3/2}(\mathbb{R}_T)$ and, moreover, the functions $R_t R_x^{-1/2}$, $R_x^{3/2}$, and $R_{xx}R_x^{-1/2}$ are elements of $L^2(\mathbb{R}_T)$. Thus, the identity

$$h(A(R)R_x)^{1/2} = \frac{R_t - (A(R)R_x)_x}{(A(R)R_x)^{1/2}} \tag{2.9}$$

holds in $L^2(\mathbb{R}_T)$. Finally, putting (2.9), (2.8), and (2.6) together, we end up with

$$I_{\iota,\mu}(\gamma) = \frac{1}{4}\big\|\widetilde{h}_x\big(A(R^{(\gamma)})R_x^{(\gamma)}\big)^{1/2}\big\|_{L^2(\mathbb{R}_T)} = J_{\iota,\mu}(\gamma)$$

of (1.8), as claimed. □

Corollary 2.3 allows us to replace the function $I_{1,\rho_0}(\cdot)$ of the large deviations upper bound for Theorem 1.4 (see (2.13)), by $J_{1,\rho_0}(\cdot)$ of the corresponding lower bound (see Proposition 2.7). The task of proving such a lower bound is further simplified thanks to the next proposition, for which we first introduce some relevant notations.

DEFINITION 2.4. Let $\mathscr{G}$ denote the subset of $\{\gamma \in \mathscr{C} : \widetilde{J}(\gamma) < \infty\}$ for which $R := R^{(\gamma)} \in C_b^\infty(\mathbb{R}_T)$, R_x is strictly positive, and (1.10) holds pointwise for some $h \in C_b(\mathbb{R}_T)$ with $x \mapsto h(t,x)$ uniformly Lipschitz-continuous on $\mathbb{R}_T$.

PROPOSITION 2.5. *Suppose that Assumption* 1.2(a), (b), *and* (d) *hold, and that* $J_{1,\mu}(\gamma) < \infty$ *for some* $\mu \in M_1^{(1)}(\mathbb{R})$. *Then there exist* $\gamma^{\ell,\epsilon} = (1-\ell^{-1})\gamma^\epsilon + \ell^{-1}\widehat{\mu}^\epsilon \in \mathscr{G}$ *for some* $\widehat{\mu}^\epsilon \in M_1^{(0)}(\mathbb{R})$ *and* $\gamma^\epsilon \in \mathscr{C}$ *such that*

$$\sup_\epsilon \int_{\mathbb{R}} |x|^2 \,\mathrm{d}\gamma^\epsilon(0) < \infty, \tag{2.10}$$

$$\lim_{\epsilon \to 0} \limsup_{\ell \to \infty} \; d(\gamma^{\ell,\epsilon}, \gamma) = 0, \tag{2.11}$$

$$\limsup_{\epsilon \to 0} \limsup_{\ell \to \infty} \widetilde{J}(\gamma^{\ell,\epsilon}) = J_{1,\mu}(\gamma) < \infty. \tag{2.12}$$

We proceed to state our basic local large deviations bounds.

PROPOSITION 2.6. *With $B(\gamma, \delta)$ denoting the open ball of radius $\delta > 0$ centered at arbitrary $\gamma \in \mathscr{C}$, under Assumption* 1.2(a)–(c) *we have the local large deviations upper bound*

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\rho^N \in B(\gamma, \delta)) \le -I_{\iota_\star, \rho_0}(\gamma). \tag{2.13}$$

Moreover, the sequence $\{\rho^N : N \in \mathbb{N}\}$ is exponentially tight in the sense that for any $M < \infty$ there exists a compact set $K_M \subset \mathscr{C}$ for which

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\rho^N \notin K_M) \le -M. \tag{2.14}$$

PROPOSITION 2.7. *Under Assumption* 1.2(a)–(c) *with $\iota_\star = 1$ and $\{\gamma^{\ell,\epsilon}\} \subset \mathscr{G}$ of Proposition* 2.5*:*

(a) *Assumption* 1.2(c) *holds for $\rho_0^{\ell,\epsilon} = \gamma^{\ell,\epsilon}(0)$, $\iota_\star = 0$, and deterministic initial empirical measures $\{\rho^{N,\ell,\epsilon}(0) : N \in \mathbb{N}\}$ such that*

$$\limsup_{\epsilon \to 0} \limsup_{\ell \to \infty} \limsup_{N \to \infty} \; d(\rho^N, \rho^{N,\ell,\epsilon}) = 0. \tag{2.15}$$

(b) *The corresponding local large deviations lower bound*

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\rho^{N,\ell,\epsilon} \in B(\gamma^{\ell,\epsilon}, \delta)) \ge -\widetilde{J}(\gamma^{\ell,\epsilon}) \tag{2.16}$$

holds for any ℓ, $\epsilon, \delta > 0$.

PROOF OF THEOREM 1.4. From [10, theorem 4.1.11] and [10, lemma 1.2.18] we conclude that Theorem 1.4 follows once we show that for $J = J_{1,\rho_0}$ and any $\gamma \in \mathscr{C}$,

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\rho^N \in B(\gamma, \delta)) \le -J(\gamma), \tag{2.17}$$

$$\lim_{\delta \downarrow 0} \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\rho^N \in B(\gamma, \delta)) \ge -J(\gamma), \tag{2.18}$$

and that the sequence $\{\rho^N : N \in \mathbb{N}\}$ is exponentially tight in the sense of (2.14). To this end, note that $\{\rho^N, N \in \mathbb{N}\}$ is exponentially tight (by Proposition 2.6), and the upper bound (2.17) is a consequence of Proposition 2.6 and Corollary 2.3. Next, it clearly suffices to establish the lower bound (2.18) when $J(\gamma) < \infty$. To this end, fixing such γ and $\gamma^{\ell,\epsilon} \in \mathscr{G}$ as in Proposition 2.5 (where $\mu = \rho_0$), we know from part (a) of Proposition 2.7 and (2.11) that, for any $\delta > 0$, $\epsilon \le \epsilon_0(\delta)$,

$\ell \geq \ell_0(\epsilon, \delta)$, if $\rho^{N,\ell,\epsilon} \in B(\gamma^{\ell,\epsilon}, \delta)$, then $\rho^N \in B(\gamma, 3\delta)$ for all N large enough. From part (b) of Proposition 2.7 we deduce that then

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\rho^N \in B(\gamma, 3\delta)) \geq -\widetilde{J}(\gamma^{\ell,\epsilon}).$$

So using (2.12) we get (2.18) by taking $\ell \to \infty$, then $\epsilon \downarrow 0$, and finally $\delta \downarrow 0$. □

3 Proof of Proposition 2.2(A) and (B)

Throughout this section, $A(\cdot)$ satisfies Assumption 1.2(a), and the function $b(\cdot)$ is uniformly bounded. Then, fixing $\iota > 0$, $\mu \in M_1(\mathbb{R})$, and $R = R^{(\gamma)}$ for $\gamma \in \mathscr{C}$ with $I_{\iota,\mu}(\gamma) < \infty$, we prove Proposition 2.2(A) in a series of three lemmas, starting with Lemma 3.1 by showing that $R_x \in L^{3/2}(\mathbb{R}_T)$, which we improve in Lemma 3.2 to L^p estimates on R_x for all $\frac{3}{2} \leq p < 3$. Finally, Lemma 3.3 establishes the uniform boundedness of the corresponding norms when $\mu \in M_1^{(0)}(\mathbb{R})$, resulting with $R_x \in L^3(\mathbb{R}_T)$.

LEMMA 3.1. *If $R = R^{(\gamma)}$ and $I_{\iota,\mu}(\gamma) = I < \infty$ for some $\mu \in M_1(\mathbb{R})$, then:*

(a) *The function $a := A(R)$ is uniformly continuous on $\mathbb{R}_T$.*

(b) *The measure $\mathrm{d}\gamma(t)\mathrm{d}t$ on $\mathbb{R}_T$ has a density with respect to the Lebesgue measure on $\mathbb{R}_T$, whose L^2 norm restricted to any strip $S_{n,r} := [0, T] \times [n - \frac{r}{4}, n + \frac{r}{4}]$ is bounded by a constant $C(T, I, r) < \infty$ (independent of $n \in \mathbb{Z}$). In particular, this density is locally square integrable, so that the weak derivative R_x exists as a function in $L^2_{loc}(\mathbb{R}_T)$.*

(c) *The weak derivative R_x is an element of $L^{3/2}(\mathbb{R}_T)$.*

PROOF.

(a) Recall that $A(\cdot)$ is assumed to be Lipschitz, so it suffices to show uniform continuity of $R = R^{(\gamma)}$ on $\mathbb{R}_T$. We further assumed that $\gamma \in \mathscr{A}_{\iota,\mu}$, hence $R \in C_b(\mathbb{R}_T)$ is uniformly continuous on compact sets. In addition, the continuity of $t \mapsto \gamma(t)$ with respect to the topology of weak convergence in $M_1(\mathbb{R})$ implies that the image $\{\gamma(t)\}_{t\in[0,T]}$ of the compact $[0, T]$ is compact in $M_1(\mathbb{R})$, hence by Prokhorov's theorem, uniformly tight. Consequently, for every $\alpha > 0$, there exists finite $M = M_\alpha$ such that

$$\sup_{\substack{t\in[0,T],\\ x\geq M}} \max(R(t, -x), 1 - R(t, x)) < \alpha, \tag{3.1}$$

extending the uniform continuity of R to all of $\mathbb{R}_T$.

(b) Setting hereafter $S_n = S_{n,2}$, it suffices to show that the inequality

$$\int_{S_n} \psi(t, x)\mathrm{d}\gamma(t)\mathrm{d}t \leq C(T, I)\|\psi\|_2 \tag{3.2}$$

holds for some constant $C(T, I) < \infty$, all $n \in \mathbb{Z}$, and any continuous $\psi : S_n \to \mathbb{R}_+$. Indeed, this implies the existence of the Radon-Nikodym derivative $R_x = \mathrm{d}\gamma(t)\mathrm{d}t/\mathrm{d}m$ on $\mathbb{R}_T$, whose $L^2(S_n)$ norm is bounded by $C(T, I)$ (by the same argument we used en route to (2.6)). Turning to proving (3.2), by definition of $I_{\iota,\mu}(\gamma)$ and the identity $\inf_{\lambda>0}\{\lambda y^2 + \lambda^{-1}z^2\} = 2|yz|$, we have that for $C_1 = 2(TI\|A\|_\infty)^{1/2}$ finite, $a(t, x) := A(R(t, x))$, and any $g \in \widetilde{\mathscr{S}}$,

$$C_1\|g_x\|_\infty \geq 2\, I^{1/2}\bigg(\int\limits_{\mathbb{R}_T} (g_x)^2 a(t, x)\mathrm{d}\gamma(t)\mathrm{d}t\bigg)^{1/2} \geq \Phi_\gamma(g). \tag{3.3}$$

Considering first $n = 0$, we use [25, theorem 2] (taking there $d = 1$ and $f(t, x) := \psi(T - t, x)$). It provides a universal finite constant C_2 and space-time kernels $k^\epsilon(t, x) = \epsilon^{-2}\xi(t/\epsilon)\xi(x/\epsilon)$, $\epsilon > 0$, for some infinitely differentiable probability density $\xi(\cdot)$ of compact support, with the following property:

> For any continuous $\psi : S_0 \mapsto \mathbb{R}_+$ there exists bounded measurable $z : \mathbb{R}_T \mapsto (-\infty, 0]$ (depending on ψ), nondecreasing in t, and supported on a larger strip $S_{0,4}$ *within which* it is convex in x such that for any $\epsilon > 0$ small enough, the smooth functions $\psi^\epsilon = \psi * k^\epsilon$ and $z^\epsilon = z * k^\epsilon$ satisfy on the intermediate strip $S_{0,3}$ the inequalities

$$\forall c \geq 0 : c^{1/2}\psi^\epsilon \leq C_2\big(z^\epsilon_t + c\, z^\epsilon_{xx}\big), \tag{3.4}$$

$$\frac{1}{2}|z^\epsilon_x| \leq -z^\epsilon \leq C_2\,\|\psi\|_2. \tag{3.5}$$

In the preceding, the compact support of z is specified in the proof of [25, theorem 2] after [25, display (29)].

Picking a $[0, 1]$-valued truncation function $\phi \in \widetilde{\mathscr{S}}$, supported on $S_{0,3}$ with $\phi \equiv 1$ on $S_{0,2}$, we note that for each $\epsilon > 0$ as above, the nonnegative function $g = -z^\epsilon\phi$ is in $\widetilde{\mathscr{S}}$, supported on $S_{0,3}$ such that by (3.5) both $\|g\|_\infty$ and $\|g_x\|_\infty$ are bounded by $C_3\|\psi\|_2$ (for the universal constant $C_3 = (2 + \|\phi_x\|_\infty)C_2$). Consequently, applying the bound (3.3) for such choice of g, we deduce that

$$\begin{aligned} C_1C_3\|\psi\|_2 \geq \Phi_\gamma(g) \geq -\int\limits_{S_{0,3}} &(g_t + ag_{xx})\mathrm{d}\gamma(t)\mathrm{d}t \\ &- T\|b(R)\|_\infty\|g_x\|_\infty - \|g(0, \cdot)\|_\infty. \end{aligned}$$

Next, with ϕ_t, ϕ_x, and ϕ_{xx} uniformly bounded by some universal constant C_4, it follows from Leibniz's rule and (3.5) that

$$\begin{aligned} |(g_t + ag_{xx}) + (z^\epsilon_t + az^\epsilon_{xx})\phi| &= |z^\epsilon(\phi_t + a\phi_{xx}) + 2a\phi_x z^\epsilon_x| \\ &\leq C_2C_4(1 + 5\|a\|_\infty)\|\psi\|_2, \end{aligned}$$

out of which we deduce by simple algebra that

$$C_5\|\psi\|_2 \geq \int_{S_{0,3}} (z_t^\epsilon + a z_{xx}^\epsilon)\phi \,\mathrm{d}\gamma(t)\mathrm{d}t \tag{3.6}$$

for finite C_5 depending only on T, $\|b\|_\infty$, $\|A\|_\infty$, and the constants C_i, $i = 1,\dots,4$. With z nondecreasing in t, so are $z^\epsilon = z * k^\epsilon$. Furthermore, as $\xi(\cdot)$ has compact support, the convexity of z in x within $S_{0,4}$ implies that z^ϵ is convex in x on $S_{0,3}$ provided that $\epsilon > 0$ is small enough. Thus, both z_t^ϵ and z_{xx}^ϵ are nonnegative, so considering (3.4) for the strictly positive $c = \underline{a}$ of Assumption 1.2(a) and recalling that $\phi \equiv 1$ on $S_0 = S_{0,2}$ (with $\phi \geq 0$ elsewhere), we have from (3.6) that

$$\begin{aligned} C_2C_5\|\psi\|_2 \geq C_2 \int_{S_0} \big(z_t^\epsilon + a z_{xx}^\epsilon\big)\phi \,\mathrm{d}\gamma(t)\mathrm{d}t &\geq \int_{S_0} C_2(z_t^\epsilon + c z_{xx}^\epsilon)\mathrm{d}\gamma(t)\mathrm{d}t \\ &\geq c^{1/2} \int_{S_0} \psi^\epsilon \,\mathrm{d}\gamma(t)\mathrm{d}t. \end{aligned}$$

With $\psi \in C_b(S_0)$, clearly $\psi^\epsilon \to \psi$ uniformly on S_0 when $\epsilon \downarrow 0$, leading to (3.2) for $n = 0$ and the universal finite constant $C(T, I) = C_2C_5c^{-1/2}$, which depends only on T, I, and the functions $b(\cdot)$ and $A(\cdot)$ on $[0, 1]$.

To extend (3.2) to other values of $n \in \mathbb{Z}$, let $\mu_n(\cdot) := \mu(\cdot + n)$ and note that the path $\gamma_n(\cdot,\cdot) = \gamma(\cdot,\cdot + n) \in \mathscr{A}_{\iota,\mu_n}$ if and only if $\gamma \in \mathscr{A}_{\iota,\mu}$. Setting next $g_n(\cdot,\cdot) = g(\cdot,\cdot + n)$, we clearly have that $(\gamma_n, \mathscr{R}^{\gamma_n} g_n)(s)$ is independent of n for each $s \in [0, T]$. Hence, $I_{\iota,\mu_n}(\gamma_n) = I$ and, since to any nonnegative $\psi \in C_b(S_n)$ corresponds nonnegative $\psi_n(\cdot,\cdot) = \psi(\cdot,\cdot + n) \in C_b(S_0)$, by the preceding proof:

$$\int_{S_n} \psi(t,x)\mathrm{d}\gamma(t)\mathrm{d}t = \int_{S_0} \psi_n(t,x)\mathrm{d}\gamma_n(t)\mathrm{d}t \leq C(T,I)\|\psi_n\|_2 = C(T,I)\|\psi\|_2$$

(as C is independent of the initial measure). This completes the proof of part (b).

(c) Fixing $n \in \mathbb{Z}$ we already know that $R_x \in L^p(S_n)$ for $p = 1, 2$. Further, upon applying the Cauchy-Schwarz inequality with respect to Lebesgue measure on S_n, we have that

$$\int_{S_n} R_x^{3/2} \,\mathrm{d}m \leq \left(\int_{S_n} R_x \,\mathrm{d}m\right)^{1/2} \left(\int_{S_n} R_x^2 \,\mathrm{d}m\right)^{1/2}.$$

We have shown in part (b) that the rightmost term is bounded uniformly in n, so our claim that the left side is summable over $n \in \mathbb{Z}$ follows from the finiteness of $\sum_{|n|\geq 1} \kappa_n$, with $\kappa_n := (\int_{S_n} R_x \,\mathrm{d}m)^{1/2}$. Next, taking $\iota > 0$ as in the given moment condition (1.6), we get by Cauchy-Schwarz that for $c_1 := \sum_{|n|\geq 1} |n|^{-(1+\iota)}$ and

$c_2 := \sup\{|n/y|^{1+\iota} : y \in S_n, |n| \geq 1\}$ finite,

$$\Big(\sum_{|n|\geq 1} \kappa_n\Big)^2 \leq c_1 \sum_{|n|\geq 1} |n|^{1+\iota}\kappa_n^2 \leq c_1 c_2 \sum_{|n|\geq 1} \int_{S_n} |y|^{1+\iota} R_x(t,y) \mathrm{d}m(t,y)$$

$$\leq c_1 c_2 \int_{\mathbb{R}_T} |y|^{1+\iota} \,\mathrm{d}\gamma(t)\mathrm{d}t < \infty$$

(see (1.6) for the rightmost inequality). □

LEMMA 3.2. *If $R = R^{(\gamma)}$ and $I_{\iota,\mu}(\gamma) < \infty$, then R_x exists as an element of $L^p(\mathbb{R}_T)$ for all $\frac{3}{2} \leq p < 3$.*

PROOF. We fix $\frac{3}{2} \leq p < 3$ and set $\frac{3}{2} < q \leq 3$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For any given nonnegative $f \in \widetilde{\mathscr{S}}$, we consider the backward Cauchy problem

$$u_t + a(t,x)u_{xx} + f(t,x)u = 0, \quad u(T,\cdot) = 1 \tag{3.7}$$

with $a(t,x) := A(R(t,x))$. While $a(t,x) \in C_b(\mathbb{R}_T)$ is not differentiable in x, recall from part (a) of Lemma 3.1 that it is uniformly continuous and bounded away from 0. Considering [26, theorem 2.1] for $w := u - 1$, such $a(\cdot,\cdot)$ and any $f \in L^r(\mathbb{R}_T)$, we see that (3.7) has a weak solution, for which $w = u - 1$ is an element of $W_r^{1,2}(\mathbb{R}_T)$. Further, here $f \in L^q(\mathbb{R}_T) \cap L^2(\mathbb{R}_T) \cap L^3(\mathbb{R}_T)$, from which it follows that the norm bounds of [26, theorem 4.1] can be refined to an estimate on the norms in $L^q(\mathbb{R}_T) \cap L^2(\mathbb{R}_T) \cap L^3(\mathbb{R}_T)$. From the latter estimate, we conclude upon applying the method of continuity (see, e.g., [30, sec. III.2]), that for $f \in \widetilde{\mathscr{S}}$ the problem (3.7) has a weak solution, for which $u - 1$ is an element of $W_q^{1,2}(\mathbb{R}_T) \cap W_2^{1,2}(\mathbb{R}_T) \cap W_3^{1,2}(\mathbb{R}_T)$.

Next, let $Y(\cdot)$ denote the canonical process having the law P of a diffusion with generator $a(t,x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ (which exists thanks to uniform continuity and the boundedness of a from above and below). Applying Itô's formula in the form of the last identity in [27, chap. 10, theorem 1] (again using the boundedness of a from above and below), we obtain the stochastic representation of such a weak solution of (3.7),

$$u(t,x) = \mathbb{E}^P\bigg[\exp\bigg(\int_t^T f(s,Y(s))\mathrm{d}s\bigg) \,\bigg|\, Y(t) = x\bigg]. \tag{3.8}$$

The latter representation implies that $u \geq 1$, and by Portenko's lemma (see [35, inequality (6)], note that it only relies on the standard heat kernel estimate for the diffusion with law P and that $q > \frac{3}{2}$ throughout this proof), there is a nondecreasing function $G_q : [0,\infty) \to [1,\infty)$ depending only on $\frac{3}{2} < q \leq 3$ (and *not on f*) such that

$$1 \leq u(t,x) \leq G_q\big(\|f\|_{L^q(\mathbb{R}_T)}\big), \quad (t,x) \in \mathbb{R}_T.$$

Next, observe that the nonnegative $\mathsf{v} := \log u$ inherits the bound

$$0 \leq \mathsf{v}(t,x) \leq G_q(\|f\|_{L^q(\mathbb{R}_T)}), \quad (t,x) \in \mathbb{R}_T. \tag{3.9}$$

Recall that $u_t, u_{xx} \in L^3(\mathbb{R}_T)$, and with $u \in W_2^{1,2}(\mathbb{R}_T)$ further $u_x \in L^6(\mathbb{R}_T)$ (by the parabolic Sobolev inequality in the form of [29, chap. II, lemma 3.3]), which for positive u bounded away from 0, imply also that $\mathsf{v}_t, \mathsf{v}_{xx} \in L^3(\mathbb{R}_T)$ and $\mathsf{v}_x \in L^6(\mathbb{R}_T)$ (with $\mathsf{v}_{xx} + \mathsf{v}_x^2 = u_{xx}/u$). From this it follows in turn that m-a.e. on $\mathbb{R}_T$ such v and its generalized derivatives satisfy the backward equation

$$\mathsf{v}_t + a(t,x)\mathsf{v}_{xx} + a(t,x)\mathsf{v}_x^2 + f(t,x) = 0, \quad \mathsf{v}(T,\cdot) = 0. \tag{3.10}$$

Recall that while proving Corollary 2.3 we found in (2.8) a function h whose $L^2(\mathbb{R}_T, aR_x \mathrm{d}m)$ norm is bounded by $C_6 := 2I^{1/2} + \|b\|_\infty (T/\underline{a})^{1/2}$ (due to (2.6) and the assumed bounds on $b(\cdot)$ and $A(\cdot) \geq \underline{a}$), so that, for any $g \in \widetilde{\mathscr{S}}$,

$$(\gamma, g)(T) - (\gamma, g)(0) = \int_{\mathbb{R}_T} (g_t + ag_{xx} - ag_x h) R_x \,\mathrm{d}m \tag{3.11}$$

(consider (2.7)). With $R_x \in L^{3/2}(\mathbb{R}_T)$ (by part (c) of Lemma 3.1) and Hölder's inequality

$$\int_{\mathbb{R}_T} |\varphi| R_x \mathrm{d}m \leq \left(\int R_x^{3/2} \,\mathrm{d}m \right)^{2/3} \left(\int |\varphi|^3 \,\mathrm{d}m \right)^{1/3},$$

we deduce that v_t, $a\mathsf{v}_{xx}$, and $a\mathsf{v}_x^2$ are integrable with respect to $R_x \,\mathrm{d}m$ (as is ah^2). Thus, with v bounded, we get from (3.11) upon approximating v in a suitable (mixed) norm by functions $g_k \in \widetilde{\mathscr{S}}$ that

$$(\gamma, \mathsf{v})(T) - (\gamma, \mathsf{v})(0) = \int_{\mathbb{R}_T} (\mathsf{v}_t + a\mathsf{v}_{xx} - a\mathsf{v}_x h) R_x \,\mathrm{d}m.$$

The latter identity, in combination with (3.10), results in

$$\begin{aligned} \int_{\mathbb{R}_T} f R_x \,\mathrm{d}m &= -\int_{\mathbb{R}_T} (\mathsf{v}_x h + \mathsf{v}_x^2) a R_x \,\mathrm{d}m + (\gamma, \mathsf{v})(0) \\ &\leq \frac{1}{4} \int_{\mathbb{R}_T} h^2 a R_x \,\mathrm{d}m + \sup_{(t,x)\in\mathbb{R}_T} \mathsf{v}(t,x) \leq \frac{1}{4} C_6^2 + G_q(\|f\|_{L^q(\mathbb{R}_T)}), \end{aligned}$$

where the first inequality is merely the nonnegativity of the $L^2(\mathbb{R}_T, aR_x \mathrm{d}m)$ norm of $\mathsf{v}_x + h/2$, and the second follows from (3.9). With C_6 independent of f, we have that the linear functional $f \mapsto \int_{\mathbb{R}_T} f R_x \,\mathrm{d}m$ is bounded on $\widetilde{\mathscr{S}}$ with respect to the $L^q(\mathbb{R}_T)$ norm, hence $R_x \in L^p(\mathbb{R}_T)$ for $1/p = 1 - 1/q$, as claimed. $\square$

LEMMA 3.3. *Suppose $R = R^{(\gamma)}$ and $I_{\iota,\mu}(\gamma) = I < \infty$ for some $\mu \in M_1^{(0)}(\mathbb{R})$. Then $R_x \in L^3(\mathbb{R}_T)$.*

PROOF. Recall from Lemma 3.2 that $R_x \in L^p(\mathbb{R}_T)$ for any $p \in [\frac{3}{2}, 3)$. Hence, $R_x \in L^3(\mathbb{R}_T)$, provided $p \mapsto \|R_x\|_{L^p(\mathbb{R}_T)}$ is uniformly bounded over such p, as we prove here.

Step 1. Recall that $a(t,x) = A(R)$ and $\int_{\mathbb{R}_T}(g_t R_x + g_{xt} R)\mathrm{d}m = 0$ (integration by parts in x). When substituted in (3.11), these yield that the path of CDFs $R = R^{(\gamma)}$ is a generalized solution of the Cauchy problem

$$R_t - (A(R)R_x)_x = hA(R)R_x, \quad R(0,\cdot) = F_\mu, \tag{3.12}$$

with respect to the collection $\widetilde{\mathscr{S}}_x := \{g_x : g \in \widetilde{\mathscr{S}}\}$ of test functions, for some function h whose $L^2(\mathbb{R}_T, aR_x\,\mathrm{d}m)$ norm is bounded by finite $C_6 = C_6(T,I)$. That is,

$$\int_{\mathbb{R}} (g_x R)(T,x)\mathrm{d}x - \int_{\mathbb{R}} (g_x R)(0,x)\mathrm{d}x = \int_{\mathbb{R}_T} (g_{xt} R - g_{xx} a R_x + g_x h a R_x)\mathrm{d}m \tag{3.13}$$

for any $g \in \widetilde{\mathscr{S}}$.

Next, applying for $f = A(R)R_x$ Hölder's inequality in the form of

$$\int_{\mathbb{R}_T} |h|^q f^q\,\mathrm{d}m \le \left(\int_{\mathbb{R}_T} |h|^2 f\,\mathrm{d}m\right)^{q/2} \left(\int_{\mathbb{R}_T} f^p\,\mathrm{d}m\right)^{q/(2p)} \tag{3.14}$$

with $\frac{3}{2} \le p < 3$ and $\frac{q}{2} + \frac{q}{2p} = 1$, namely $q := \frac{2p}{p+1} \in [\frac{6}{5}, \frac{3}{2})$, we deduce that $hA(R)R_x \in L^q(\mathbb{R}_T)$. Thus, by the regularity theory for the heat equation (see inequalities (3.1) and (3.2) in [29, chap. IV] or [26, theorem 2.1]), we have that the function

$$V(t,x) := \int_0^t \int_{\mathbb{R}} (hA(R)R_x)(s,y)\varphi(t-s,x-y)\mathrm{d}y\,\mathrm{d}s,$$

obtained by convolving with the heat kernel $\varphi(t,x) = (4\pi t)^{-1/2}\exp(-x^2/4t)$, is a generalized solution of the auxiliary Cauchy problem

$$V_t - V_{xx} = hA(R)R_x, \quad V(0,\cdot) = 0 \tag{3.15}$$

(with respect to the collection $\widetilde{\mathscr{S}}_x$). In addition,

$$\|V\|_{W_q^{1,2}(\mathbb{R}_T)} \le C_7 \|hA(R)R_x\|_{L^q(\mathbb{R}_T)},$$

where $C_7 < \infty$ is a uniform constant (which in particular does not depend on q as long as q belongs to a compact interval). Thus, $V \in W_q^{1,2}(\mathbb{R}_T)$ and, due to the parabolic Sobolev inequality (in the form of [29, chap. II, lemma 3.3]), also $V, V_x \in L^r(\mathbb{R}_T)$ for any $q \le r \le p'$ where $p' := \left(\frac{1}{q} - \frac{1}{3}\right)^{-1} = \frac{6p}{3+p}$. Further, the constants in this parabolic Sobolev inequality can be chosen uniformly over p in any given compact interval; hence for some uniform $C_8 < \infty$ and *all* $p \in [\frac{3}{2}, 3)$,

$$\|V_x\|_{L^{p'}(\mathbb{R}_T)} \le C_8 \|h\,A(R)R_x\|_{L^q(\mathbb{R}_T)} \tag{3.16}$$

where $q = \frac{2p}{p+1}$ and $p' = \frac{6p}{3+p}$. Similarly, the function

$$Z(t,x) := \int_{\mathbb{R}} F_\mu(y)\varphi(t, x-y)\mathrm{d}y,$$

is a classical solution of the initial value problem

$$Z_t - Z_{xx} = 0, \quad Z(0,\cdot) = F_\mu. \tag{3.17}$$

Since $F_\mu(y) = \int_{-\infty}^{y} \theta(z)\mathrm{d}z$ for $\theta := \frac{\mathrm{d}\mu}{\mathrm{d}x}$, clearly Z_x is given by the convolution (in space) of θ and the heat kernel $\varphi(t,\cdot)$. Consequently, by Fubini's theorem and Young's inequality, for any $p, q \geq 1$ such that $1/p + 1/q = 1/r + 1$,

$$\|Z_x\|^r_{L^r(\mathbb{R}_T)} = \int_0^T \|Z_x(t,\cdot)\|^r_{L^r(\mathbb{R})}\mathrm{d}t \leq \|\theta\|^r_{L^q(\mathbb{R})} \int_0^T \|\varphi(t,\cdot)\|^r_{L^p(\mathbb{R})}\mathrm{d}t.$$

By assumption $\mu \in M_1^{(0)}(\mathbb{R})$, so $\theta \in L^1(\mathbb{R}) \cap L^{q_0}(\mathbb{R})$ for some $q_0 > 1$, and hence the norms $\|\theta\|_{L^q(\mathbb{R})}$, $1 \leq q \leq q_0$, are uniformly bounded. Further, $\|\varphi(t,\cdot)\|_{L^p(\mathbb{R})} \leq t^{-(p-1)/(2p)}$ for all $t > 0$ and $p \geq 1$. Considering $q = q_0 \wedge r$, with $p \geq 1$ such that for $1 \leq r \leq 3$ the value of $r(p-1)/(2p)$ is bounded away from 1, we conclude that $\|Z_x\|_{L^r(\mathbb{R}_T)}$, $1 \leq r \leq 3$, are uniformly (in r) bounded by some finite $C_9 = C_9(T, \|\theta\|_{q_0})$.

From (3.12), (3.15), and (3.17) it follows by a direct computation that $U := R - V - Z$ is a generalized solution of the Cauchy problem

$$U_t - (A(R)U_x)_x = [(A(R)-1)V_x + (A(R)-1)Z_x]_x, \quad U(0,\cdot) = 0, \tag{3.18}$$

with respect to test functions in $\widetilde{\mathscr{S}_x}$ (interpreted via integration by parts in t and x, similarly to what we have done in (3.13)).

Step 2. We proceed to obtain the uniform-in-p bounds on L^p norms of U_x (and thereby those of R_x) out of the bounds we already have for V_x and Z_x. To this end, recalling part (a) of Lemma 3.1, refining the norm estimates in [26, theorem 6.2] to estimates in $L^p(\mathbb{R}_T) \cap L^{p'}(\mathbb{R}_T)$ with $p' = \frac{6p}{3+p}$, and applying the method of continuity (see, e.g., [30, sec. III.2]), we deduce the existence of a solution $\widehat{U}$ of the problem (3.18) in the space $\mathscr{H}_p(T) \cap \mathscr{H}_{p'}(T)$ defined in [26]. In particular, $\widehat{U} \in L^{p'}(\mathbb{R}_T)$, $\widehat{U}_x \in L^{p'}(\mathbb{R}_T)$, and

$$\|\widehat{U}\|_{L^{p'}(\mathbb{R}_T)} + \|\widehat{U}_x\|_{L^{p'}(\mathbb{R}_T)} \leq C_{10}\big(\|V_x\|_{L^{p'}(\mathbb{R}_T)} + \|Z_x\|_{L^{p'}(\mathbb{R}_T)}\big) \tag{3.19}$$

with $p' = \frac{6p}{3+p}$ and where the finite constant $C_{10} = C_{10}(T, \|A\|_\infty, \underline{a})$ can be chosen uniformly for all $2 \leq p' < 3$.

We show in Step 3 below that $U = \widehat{U}$ Lebesgue a.e. on $\mathbb{R}_T$. Thus, $U_x \in L^{p'}(\mathbb{R}_T)$ and consequently also $R_x = U_x + V_x + Z_x \in L^{p'}(\mathbb{R}_T)$. Hence, combining the norm bounds (3.16) and (3.19) with Hölder's inequality (3.14) yields the estimate

$$\|R_x\|_{L^{p'}(\mathbb{R}_T)} \le C_{11}\|R_x\|^{1/2}_{L^p(\mathbb{R}_T)} + C_{12} \tag{3.20}$$

for the finite constants $C_{11} = (1+C_{10})C_8(C_6\|A\|_\infty)^{1/2}$ and $C_{12} = (1+C_{10})C_9$ (both independent of $\frac{3}{2} \le p < 3$). Since $p' \ge p > 1$, by Jensen's inequality (for the convex function $x^{(p'-1)/(p-1)}$ and the probability measure $T^{-1}R_x\,\mathrm{d}m$ on $\mathbb{R}_T$), we deduce that for any $p \in [\frac{3}{2}, 3)$

$$\|R_x\|^{r(p)}_{L^p(\mathbb{R}_T)} \le C_{13}\|R_x\|_{L^{p'}(\mathbb{R}_T)},$$

where $r(p) = \frac{p(p'-1)}{p'(p-1)} \ge 1$ and $C_{13} = \max(1, T^{1/2})$ is finite. Combining this with (3.20) we conclude that

$$\|R_x\|_{L^p(\mathbb{R}_T)} \le \max\Big\{1, C_{13}\Big(C_{11}\,\|R_x\|^{1/2}_{L^p(\mathbb{R}_T)} + C_{12}\Big)\Big\}.$$

As explained before, having $\|R_x\|_{L^p(\mathbb{R}_T)}$ bounded, uniformly over $\frac{3}{2} \le p < 3$, yields that $R_x \in L^3(\mathbb{R}_T)$ (whose L^3-norm is bounded by some $C(T, I)$ finite).

Step 3. Turning to show that Lebesgue a.e. $U = \widehat{U}$ on $\mathbb{R}_T$, we start by verifying that $R - Z \in L^p(\mathbb{R}_T)$. Indeed, as $|R - Z| \le 1$, it suffices to check this for $p = 1$, in which case by the triangle inequality

$$\begin{aligned}\int_{\mathbb{R}_T} |R - Z|\mathrm{d}m &\le \int_{\mathbb{R}_T^+} (1 - R)\mathrm{d}m + \int_{\mathbb{R}_T^+} (1 - Z)\mathrm{d}m + \int_{\mathbb{R}_T^-} R\,\mathrm{d}m + \int_{\mathbb{R}_T^-} Z\,\mathrm{d}m \\ &= \int_{\mathbb{R}_T} |x|R_x\,\mathrm{d}m + \int_{\mathbb{R}_T} |x|Z_x\,\mathrm{d}m, \end{aligned} \tag{3.21}$$

where $\mathbb{R}_T^+ = [0, T] \times \mathbb{R}_+$, $\mathbb{R}_T^- = [0, T] \times \mathbb{R}_-$, and the last equality applies since $R(t, \cdot)$ and $Z(t, \cdot)$ are CDFs having densities R_x and Z_x with respect to Lebesgue measure on $\mathbb{R}_T$. Since $\gamma \in \mathscr{A}_{\iota,\mu}$, the first term on the right side of (3.21) is finite (see (1.6)), whereas the second term amounts to $\int_0^T \mathbf{E}[|Y + W(2t)|]\mathrm{d}t$ for Y of law μ, independently of the standard Brownian motion $\{W\}$. By the triangle inequality, the latter term is at most $\frac{1}{3}(2T)^{3/2} + T\int_{\mathbb{R}} |x|\mathrm{d}\mu$, hence finite (since $\mu \in M_1^{(0)}(\mathbb{R})$).

Now, with $R - Z \in L^p(\mathbb{R}_T)$, and having seen already in Step 1 that $V \in L^p(\mathbb{R}_T)$ (by the parabolic Sobolev inequality for $V \in W_q^{1,2}(\mathbb{R}_T)$), we conclude that $U = R - Z - V \in L^p(\mathbb{R}_T)$. Similarly, $U_x \in L^p(\mathbb{R}_T)$, since $R_x \in L^p(\mathbb{R}_T)$ (from Lemma 3.2), and we have already established in Step 1 that $Z_x \in L^p(\mathbb{R}_T)$ and $V_x \in L^p(\mathbb{R}_T)$.

Recall that $h \in L^1(R_x\,\mathrm{d}m)$, so $V \in L^1(\mathbb{R}_T)$, and hence $U \in L^1(\mathbb{R}_T)$ as well. Now, fixing $g \in \widetilde{\mathscr{S}}$ we let

$$f(t,x) = -\int_x^\infty g(t,y)\mathrm{d}y, \quad (t,x) \in \mathbb{R}_T, \tag{3.22}$$

and claim that, for Lebesgue almost every $0 \le t_1 < t_2 \le T$,

$$\begin{aligned}\int_{\mathbb{R}} f(t_2,x)U(t_2,x)\mathrm{d}x - \int_{\mathbb{R}} f(t_1,x)U(t_1,x)\mathrm{d}x - \int_{t_1}^{t_2}\int_{\mathbb{R}} Uf_t\,\mathrm{d}m = \\ -\int_{t_1}^{t_2}\int_{\mathbb{R}} [A(R)U_x + (A(R)-1)(V_x+Z_x)]f_x\,\mathrm{d}m.\end{aligned} \tag{3.23}$$

Indeed, from the weak formulation of the PDE in (3.18) we have (3.23) when $f \in \widetilde{\mathscr{S}}_x$. This in turn extends to all f as in (3.22), by the uniform joint approximation on compacts of the continuously differentiable and bounded (f, f_t, f_x) by $(\widetilde{g}, \widetilde{g}_t, \widetilde{g}_x)$ for some $\widetilde{g} \in \widetilde{\mathscr{S}}_x$ (taking such a compact large enough that the contribution to (3.23) from its complement is as small as desired, which we can do since $A(R)U_x + (A(R)-1)(V_x+Z_x) \in L^p(\mathbb{R}_T)$ and $U \in L^1(\mathbb{R}_T)$). Next, integrating by parts in space on the left side of (3.23), we conclude that the function $W(t,x) = \int_{-\infty}^x U(t,y)\mathrm{d}y$ is a generalized solution of the Cauchy problem

$$W_t = A(R)W_{xx} + (A(R)-1)(V_x+Z_x), \quad W(0,\cdot) = 0, \tag{3.24}$$

on $\mathbb{R}_T$ (with respect to test functions from $\widetilde{\mathscr{S}}$). This, $W_{xx} = U_x \in L^p(\mathbb{R}_T)$, $V_x \in L^p(\mathbb{R}_T)$, and $Z_x \in L^p(\mathbb{R}_T)$ imply $W_t \in L^p(\mathbb{R}_T)$. Thus, in view of [26, norm estimate (6.1)], we have that $U \in \mathscr{H}_p(\mathbb{R}_T)$, so that $U = \widehat{U}$ Lebesgue almost everywhere by [26, theorem 2.4]. □

PROOF OF PROPOSITION 2.2(B). From the proof of Lemma 3.3 we recall that $R - Z, Z_x, R_x \in L^r(\mathbb{R}_T)$ for all $r \in [1,3]$ and $V \in W_q^{1,2}(\mathbb{R}_T)$ for all $q \in [\frac{6}{5}, \frac{3}{2}]$ (where, since $R_x \in L^3(\mathbb{R}_T)$, the argument leading to (3.16) now applies also for $p = 3$ and $q = \frac{3}{2}$). Consequently, $V_t, V_{xx} \in L^q(\mathbb{R}_T)$ for any $q \in [\frac{6}{5}, \frac{3}{2}]$ and, by the parabolic Sobolev inequality (in the form of [29, chap. II, lemma 3.3]), also $V, V_x \in L^{p'}(\mathbb{R}_T)$ for any $p' = (\frac{1}{q} - \frac{1}{3})^{-1} \in [\frac{3}{2}, 3]$. Since $U = R - Z - V$, this in turn implies that $U, U_x \in L^p(\mathbb{R}_T)$ for all $p \in [\frac{3}{2}, 3]$. Further, $Z_t = Z_{xx} \in L^q(\mathbb{R}_T)$ for all $q \in [\frac{6}{5}, \frac{3}{2}]$; hence it suffices to show that $U \in W_q^{1,2}(\mathbb{R}_T)$, as then $U_t, U_{xx} \in L^q(\mathbb{R}_T)$, implying the same for R_t and R_{xx}. To this end, rewriting (3.18) we have that U solves

$$U_t - A(R)U_{xx} = A'(R)R_x\,U_x + \big[(A(R)-1)V_x + (A(R)-1)Z_x\big]_x \tag{3.25}$$

with respect to test functions in $\widetilde{\mathscr{S}}_x$, starting at $U(0,\cdot) = 0$. For any $q \in [\frac{6}{5}, \frac{3}{2}]$ there exist $r, p \in [\frac{3}{2}, 3]$ such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$, so with $A(\cdot)$ and $A'(\cdot)$ bounded, by Hölder's inequality and the preceding integrability properties, the right side in

(3.25) belongs to $L^q(\mathbb{R}_T)$ for all $q \in [\frac{6}{5}, \frac{3}{2}]$. Thus, fixing $q \in [\frac{6}{5}, \frac{3}{2}]$ we may apply [26, theorem 2.1] to deduce that there is a function $\widetilde{U} \in W_q^{1,2}(\mathbb{R}_T) \cap W_{3/2}^{1,2}(\mathbb{R}_T)$ that satisfies

$$\widetilde{U}_t - A(R)\widetilde{U}_{xx} = A'(R)R_x\, U_x + [(A(R)-1)V_x + (A(R)-1)Z_x]_x$$

and $\widetilde{U}(0,\cdot) = 0$. In particular, $\widetilde{U}$, $\widetilde{U}_x \in L^3(\mathbb{R}_T)$. Now, we let ϕ_k, $k \in \mathbb{N}$, be a truncation sequence such that $\phi_k \in C^\infty(\mathbb{R})$, $0 \le \phi_k \le 1$, $\phi_k \equiv 1$ on $[-k,k]$, $\phi_k \equiv 0$ on $(-\infty, -k-1] \cup [k+1, \infty)$ and $\max(|\phi_k'|, |\phi_k''|) \le 2$. Next, fixing $k \in \mathbb{N}$, we set $\widetilde{\Delta} = \phi_k(U - \widetilde{U})$. Then, $\widetilde{\Delta}$ is a generalized solution of the problem

$$\widetilde{\Delta}_t - (A(R)\widetilde{\Delta}_x)_x + A'(R)R_x\widetilde{\Delta}_x = \widetilde{\psi}_k, \quad \Delta(0,\cdot) = 0, \tag{3.26}$$

with respect to test functions in $\widetilde{\mathscr{S}_x}$, where

$$\widetilde{\psi}_k = -A(R)\phi_k''(U - \widetilde{U}) - 2A(R)\phi_k'(U - \widetilde{U})_x$$

is in $L^3(\mathbb{R}_T)$. Further, $\widetilde{\psi}_k(t,x) = 0$ for all $x \in (-k,k)$, so by our choice of ϕ_k,

$$\lim_{k\to\infty} \|\widetilde{\psi}_k\|_{L^3(\mathbb{R}_T)} = 0. \tag{3.27}$$

Now, a careful reading of the proof of [29, chap. III, theorem 3.3] shows that the solution of the problem (3.26) in the space $W_2^{0,1}(\mathbb{R}_T)$ is unique and satisfies

$$\|\widetilde{\Delta}\|_{W_2^{0,1}(\mathbb{R}_T)} \le C_9 \left(\int_0^T \left(\int_{\mathbb{R}} |\widetilde{\psi}_k|^{q_1}\, \mathrm{d}x \right)^{q_2/q_1} \mathrm{d}t \right)^{1/q_2} \tag{3.28}$$

for all $q_1 \in [2,\infty]$ and $q_2 \in [2,4]$ with $\frac{1}{q_1} + \frac{2}{q_2} = 1$ provided that

$$\int_0^T \left(\int_{\mathbb{R}} |A'(R)R_x|^{q_1}\, \mathrm{d}x \right)^{q_2/q_1} \mathrm{d}t < \infty.$$

We choose $q_1 = q_2 = 3$, so that the latter condition is satisfied. In addition, by (3.27) and (3.28), the norm $\|\widetilde{\Delta}\|_{W_2^{0,1}(\mathbb{R}_T)}$ tends to 0 in the limit $k \to \infty$, and we conclude that $U = \widetilde{U} \in W_q^{1,2}(\mathbb{R}_T)$, as claimed. □

4 Proof of Proposition 2.2(C)

The proof of Proposition 2.2(C) consists of four steps. In Step 1 we convert the variational formula $I_{\iota,\mu}(\gamma) < \infty$ into the formula (4.6), which corresponds to a suitable one-dimensional reversible diffusion. Step 2 then deduces the existence of a sufficiently regular solution to the corresponding backward Cauchy problem (4.7), which enables us to employ Dirichlet form calculus for establishing in Step 3 the integrability of R_{xx}^2/R_x for $R = R^{(\gamma)}$. From the latter we deduce in Step 4 that R_t^2/R_x is integrable.

Step 1. The functions $a(t,x) = A(R)$ and $b(R)$ are uniformly bounded and our assumption that $I_{\iota,\mu}(\gamma) < \infty$ implies that $\mathrm{d}\gamma\, \mathrm{d}t$ has density $R_x \in L^3(\mathbb{R}_T)$ with respect to Lebesgue measure $\mathrm{d}m$ (by Proposition 2.2(A)). These facts imply by multiple applications of Hölder's inequality that the functional $g \mapsto \Phi_\gamma(g) - (g,g)_\gamma$ in (2.4) is continuous with respect to the norm

$$\|g\|_\infty + \|g_t\|_{L^{3/2}(\mathbb{R}_T)} + \|g_x\|_{L^{3/2}(\mathbb{R}_T)} + \|g_x\|_{L^3(\mathbb{R}_T)} + \|g_{xx}\|_{L^{3/2}(\mathbb{R}_T)}.$$

Therefore, denoting by $\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$ the subspace of all $g \in C_b(\mathbb{R}_T)$ for which $g_t,\ g_{xx} \in L^{3/2}(\mathbb{R}_T)$ and $g_x \in L^{3/2}(\mathbb{R}_T) \cap L^3(\mathbb{R}_T)$, the assumption $I_{\iota,\mu}(\gamma) < \infty$ amounts to

$$\sup_{g \in \widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)} \left[(\gamma, g)(T) - (\gamma, g)(0) - \int_0^T \big(\gamma, \mathscr{R}^\gamma g + a g_x^2\big)(t)\mathrm{d}t \right] < \infty \tag{4.1}$$

for $\mathscr{R}^\gamma$ of (2.1). Now, fixing $\psi \in C^\infty(\mathbb{R})$ such that $\lim_{|x|\to\infty} \psi(x)/|x| = 1$, $\|\psi'\|_\infty < \infty$, and $\alpha_0 := e^{-\psi(x)}\mathrm{d}x$ is a probability measure, we introduce the parabolic operator

$$\mathscr{R}^{\gamma,\psi} = \frac{\partial}{\partial t} + e^{\psi(x)} \frac{\partial}{\partial x} a(t,x) e^{-\psi(x)} \frac{\partial}{\partial x} = \mathscr{R}^\gamma + (a_x - a\psi' - b(R)) \frac{\partial}{\partial x} \tag{4.2}$$

and show next that (4.1) implies that

$$\begin{aligned} \sup_{g \in \widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)} & \Bigg[(\gamma, g)(T) - \log\Bigg(\int_{\mathbb{R}} e^{g(0,x)} \mathrm{d}\alpha_0 \Bigg) \\ & \qquad - \int_0^T \big(\gamma, \mathscr{R}^{\gamma,\psi} g + a g_x^2\big)(t)\mathrm{d}t \Bigg] < \infty. \end{aligned} \tag{4.3}$$

Indeed, by Cauchy-Schwarz we have the bound

$$\kappa_3 := \int_{\mathbb{R}_T} (b(R) - a_x + a\psi') g_x R_x \,\mathrm{d}m \le C_2 \sqrt{(g,g)_\gamma}$$

for the finite, positive

$$C_2 := \underline{a}^{-1/2}\big(\|b\|_\infty T^{1/2} + \|A'\|_\infty \|R_x\|^{3/2}_{L^3(\mathbb{R}_T)} \big) + \|\sqrt{a}\psi'\|_\infty T^{1/2}.$$

Further, the implication

$$\begin{aligned} & \forall \lambda > 0\colon\ \lambda\kappa_1 \le C_1 + \lambda^2 \kappa_2, \quad \kappa_3 \le C_2 \sqrt{\kappa_2}, \\ \Longrightarrow\ & \forall \widetilde{\lambda} > 0\colon\ \widetilde{\lambda}\kappa_1 + \widetilde{\lambda}\kappa_3 \le 2C_1 + \frac{1}{2} C_2^2 + \widetilde{\lambda}^2 \kappa_2, \end{aligned} \tag{4.4}$$

holds for all $\kappa_1, \kappa_3 \in \mathbb{R}$ and κ_2, C_1, C_2 positive. Therefore, scaling the test functions g in (4.1) and (4.3) by $\lambda > 0$ and $\widetilde{\lambda} = 2\lambda$, respectively, then considering (4.4) for $C_1 = I_{\iota,\mu}(\gamma)$, $\kappa_1 = \Phi_\gamma(g)$, and $\kappa_2 = (g,g)_\gamma$ proves that the change from $\mathscr{R}^\gamma$

to $\mathscr{R}^{\gamma,\psi}$ in (4.1) does not make the supremum infinite. Moreover, the change in value due to the terms corresponding to the initial condition is bounded by

$$\sup_{g\in\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)}\left[\int_{\mathbb{R}} g(0,x)\mathrm{d}\mu - \log\int_{\mathbb{R}} e^{g(0,x)}\,\mathrm{d}\alpha_0\right] \le H(\mu|\alpha_0)$$

(see, for example, [10, lemma 6.2.13]). The latter relative entropy is finite since

$$H(\mu|\alpha_0) = \int_{\mathbb{R}} \log\frac{\mathrm{d}\mu}{\mathrm{d}\alpha_0}\,\mathrm{d}\mu = \int_{\mathbb{R}} \log\frac{\theta}{e^{-\psi}}\,\mathrm{d}\mu \le \int_{\mathbb{R}} |\psi|\mathrm{d}\mu + \int_{\mathbb{R}} \log\theta\,\mathrm{d}\mu, \tag{4.5}$$

while $\int_{\mathbb{R}} |x|\mathrm{d}\mu < \infty$ and $\theta \in L^{q_0}(\mathbb{R})$ for some $q_0 > 1$ (by definition, for $\mu \in M_1^{(0)}(\mathbb{R})$).

Next, let $\mathscr{E}\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$ denote the collection of $u : \mathbb{R}_T \mapsto (0,\infty)$ such that $\log u = g \in \widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$. It is easy to verify that $\mathscr{E}\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$ consists of all positive $u \in \widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$ that are bounded away from 0, in terms of which (4.3) becomes

$$\begin{aligned}\sup_{u\in\mathscr{E}\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)}\Bigg[(\gamma,\log u)(T) &- \log\int_{\mathbb{R}} u(0,x)\mathrm{d}\alpha_0\\ &- \int_0^T\left(\gamma,\frac{\mathscr{R}^{\gamma,\psi}u}{u}\right)(t)\mathrm{d}t\Bigg] < \infty.\end{aligned} \tag{4.6}$$

Step 2. We claim that to every $f \in \widetilde{\mathscr{S}}$ there corresponds a u such that

$$\mathscr{R}^{\gamma,\psi}u - fu = 0, \quad u(T,\cdot) = 1, \tag{4.7}$$

where all terms of (4.7) are then in $L^{3/2}(\mathbb{R}_T)$ (by definition $u_x, R_x \in L^r(\mathbb{R}_T)$ for all $r \in [\frac{3}{2}, 3]$; hence $(a_x - a\psi')u_x \in L^{3/2}(\mathbb{R}_T)$ by the Cauchy-Schwarz inequality and the boundedness of A', A, and ψ'). Clearly, having such a solution for (4.7) amounts to finding a solution w of

$$\mathscr{R}^{\gamma,\psi}w - fw = f, \quad w(T,\cdot) = 0,$$

or, equivalently (see (4.2)), a solution of

$$w_t + (aw_x)_x - \psi' a w_x - fw = f, \quad w(T,\cdot) = 0, \tag{4.8}$$

such that $u = (w+1) \in \mathscr{E}\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$.

To this end, we employ [26, theorem 6.2] together with the method of continuity to first get a generalized solution w of (4.8) in the space $W_6^{0,1}(\mathbb{R}_T) \cap W_2^{0,1}(\mathbb{R}_T)$. Indeed, the norm bound in [26, inequality (6.3)] extends to a norm bound for functions in $\mathscr{H}_6^1 \cap \mathscr{H}_2^1$ with respect to the norms $\|\cdot\|_{\mathbb{H}_6^{-1}} + \|\cdot\|_{\mathbb{H}_2^{-1}}$ and $\|\cdot\|_{\mathbb{H}_6^1} + \|\cdot\|_{\mathbb{H}_2^1}$ defined in [26] (one only needs to add the norm bounds [26, inequality (6.3)] for $p = 6$ and $p = 2$). Applying the method of continuity (see [30, sec. III.2]) and relying on such a refined norm estimate to interpolate between the PDE (4.8) and the corresponding PDE with a smooth coefficient a, we find a solution of (4.8) in

$\mathscr{H}_6^1 \cap \mathscr{H}_2^1$, which in particular belongs to $W_6^{0,1}(\mathbb{R}_T) \cap W_2^{0,1}(\mathbb{R}_T)$. Moreover, by Hölder's inequality $a_x w_x \in L^2(\mathbb{R}_T) \cap L^{3/2}(\mathbb{R}_T)$ (since $a_x = A'(R)R_x \in L^3(\mathbb{R}_T)$ and $w_x \in L^r(\mathbb{R}_T)$ for all $r \in [2,6]$). Hence, refining the norm bounds in [26, theorem 4.1] to an estimate on the norms in $L^2(\mathbb{R}_T) \cap L^{3/2}(\mathbb{R}_T)$ and applying the method of continuity in a similar fashion, we also have a generalized solution $\widehat{w} \in W_2^{1,2}(\mathbb{R}_T) \cap W_{3/2}^{1,2}(\mathbb{R}_T)$ of the equation

$$\widehat{w}_t + a\,\widehat{w}_{xx} - a\,\psi'\widehat{w}_x - f\,\widehat{w} = -a_x\,w_x + f, \quad \widehat{w}(T,\cdot) = 0. \tag{4.9}$$

Proceeding to show that $w = \widehat{w}$, we let $\{\phi_k, k \in \mathbb{N}\}$ be the same truncation sequence as in the proof of Proposition 2.2(B), fix $k \in \mathbb{N}$ and set $\Delta := \phi_k(\widehat{w} - w)$. Then, $\Delta \in W_2^{0,1}(\mathbb{R}_T)$ is a generalized solution of

$$\Delta_t + (a\Delta_x)_x - (a_x + \psi'\,a)\Delta_x - f\,\Delta = \psi_k, \quad \Delta(T,\cdot) = 0,$$

where

$$\psi_k = \phi_k''\,a(\widehat{w} - w) + 2\,\phi_k'\,a(\widehat{w} - w)_x - \phi_k'\,\psi'\,a(\widehat{w} - w).$$

As in the derivation of (3.27), since $\max(|\phi_k'|, |\phi_k''|) \le 2\cdot\mathbb{1}_{|x|\in[k,k+1]}$, having a, ψ' uniformly bounded and $(\widehat{w} - w), (\widehat{w} - w)_x \in L^2(\mathbb{R}_T)$ implies that

$$\lim_{k\to\infty} \|\psi_k\|_{L^2(\mathbb{R}_T)} = 0. \tag{4.10}$$

Hence, writing $\Delta = \phi_k\,\widehat{w} - \phi_k\,w$, following the paragraph after the statement of [29, chap. III, theorem 3.3] and applying the energy inequality of [29, chap. III, theorem 2.1], with $r = q = 2$ and $n = 1$ (so that $\frac{2}{r} + \frac{n}{q} \le \frac{n+4}{2}$), we conclude that

$$\|\Delta\|_{L^2(\mathbb{R}_T)} \le C_3\bigg(\int_0^T \bigg(\int_{\mathbb{R}} |\psi_k|^q\,\mathrm{d}x\bigg)^{r/q}\mathrm{d}t\bigg)^{1/r} \to 0 \tag{4.11}$$

as $k \to \infty$ .Therefore, m-a.e. $w = \widehat{w}$ on $\mathbb{R}_T$. All in all, we have found a solution u to (4.7) such that $w = u - 1$ is an element of $W_2^{1,2}(\mathbb{R}_T) \cap W_{3/2}^{1,2}(\mathbb{R}_T)$, and hence also $u_x = w_x \in L^6(\mathbb{R}_T)$ (by the parabolic Sobolev inequality in the form of [29, chap. II, lemma 3.3] for $p = 6$ and $q = 2$).

It thus remains only to show that $u \in C_b(\mathbb{R}_T)$ and that u is bounded away from 0 on $\mathbb{R}_T$. To establish this we first apply [29, chap. III, theorem 5.2] to find a generalized solution $\widetilde{w}$ of (4.8) in the subspace of $W_2^{0,1}(\mathbb{R}_T)$ whose elements satisfy

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}} \widetilde{w}(t,x)^2\,\mathrm{d}x + \int_{\mathbb{R}_T} \widetilde{w}_x^2\,\mathrm{d}m < \infty. \tag{4.12}$$

Next, we apply [1, theorem 10(vi)], with the constant $\gamma > 0$ there being arbitrarily small, to conclude that $\widetilde{u} := \widetilde{w} + 1$ has to be the unique generalized solution of (4.7) in the sense of [1]. It is thus given by

$$\widetilde{u}(t,x) = \int_{\mathbb{R}} \Gamma(t,x;T,y)\mathrm{d}y \tag{4.13}$$

with Γ denoting the weak fundamental solution of (4.7), defined as in [1, sec. 6]. Now, [1, theorem C] implies that $\widetilde{u}$ is locally Hölder continuous in (t,x), and hence continuous in (t,x) on $[0,T)\times\mathbb{R}$. Putting this together with [1, theorem 10(vi)], we conclude that $\widetilde{u}$ is continuous on the whole of $\mathbb{R}_T$.

Finally, we use the heat kernel estimates on Γ from [1, theorem 7] to conclude that $\widetilde{u}$ has to be bounded between two positive constants. Therefore, all we need to show now is that $\widetilde{u}=u$ or, equivalently, $\widetilde{w}=w$. To this end, reusing the truncation sequence $\{\phi_k\}$, we fix $k\in\mathbb{N}$ and set $\widetilde{\Delta} := \phi_k(\widetilde{w}-w)$. Then $\widetilde{\Delta}\in W_2^{0,1}(\mathbb{R}_T)$ is a generalized solution of

$$\widetilde{\Delta}_t + (a\widetilde{\Delta}_x)_x - \psi' a\widetilde{\Delta}_x - f\widetilde{\Delta} = \widetilde{\psi}_k, \quad \widetilde{\Delta}(T,\cdot)=0,$$

where

$$\widetilde{\psi}_k = \phi_k'' a(\widetilde{w}-w) + 2\phi_k' a(\widetilde{w}-w)_x + \phi_k' a_x(\widetilde{w}-w) - \phi_k'\psi' a(\widetilde{w}-w).$$

The $L^2(\mathbb{R}_T)$-norm of $\phi_k' a_x(\widetilde{w}-w)$ decays as $k\to\infty$ (because $a_x\in L^3(\mathbb{R}_T)$ and $\|\widetilde{w}-w\|_{L^6(S_{k,4}\cap S_{-k,4})}$ is uniformly bounded in k). Thus, with $w,\widetilde{w},w_x,\widetilde{w}_x\in L^2(\mathbb{R}_T)$ we deduce as in the derivation of (4.10) that

$$\lim_{k\to\infty}\|\widetilde{\psi}_k\|_{L^2(\mathbb{R}_T)} = 0.$$

This implies, as in the derivation of (4.11), that $\|\widetilde{\Delta}\|_{L^2(\mathbb{R}_T)}\to 0$ as $k\to\infty$, and consequently that m-a.e. $w=\widetilde{w}$ on $\mathbb{R}_T$.

Step 3. In view of (4.7), we get from (4.6) that

$$\infty > \sup_{f\in\mathscr{S}}\left[-\log\int_{\mathbb{R}} u^{(f)}(0,x)\mathrm{d}\alpha_0 - \int_0^T (\gamma,f)(t)\mathrm{d}t\right],$$

where $u^{(f)}\in\mathscr{E}\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$ satisfies (4.7). We then deduce that, for some $C\in(0,\infty)$,

$$\infty > \sup_{g\in\mathscr{S}}\left[-\int_0^T(\gamma, g_x + Cg^2)(t)\mathrm{d}t\right] = \sup_{g\in\mathscr{S}}\int_{\mathbb{R}_T}(gR_{xx} - Cg^2R_x)\mathrm{d}m \tag{4.14}$$

by showing that the $L^2(\alpha_0)$-norm of $u^{(f)}(0,\cdot)$ (and so also $\log\int_{\mathbb{R}}u^{(f)}(0,x)\mathrm{d}\alpha_0$) is uniformly bounded over all $f=g_x+Cg^2$ with $g\in\mathscr{S}$. To this end, recall from (4.2) that $\mathscr{R}^{\gamma,\psi}v = v_t + \mathscr{L}v$ for $v\in\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$ and the bounded linear operator

$$\mathscr{L}v = e^{\psi}(ae^{-\psi}v_x)_x : \widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)\mapsto L^{3/2}(\mathbb{R}_T).$$

We further set $\mathscr{L}_t\phi = e^{\psi}(a(t,x)e^{-\psi}\phi')_x$ for $\phi \in C_c^\infty(\mathbb{R})$. Then, considering positive $\phi_k \in \widetilde{\mathscr{S}}$ that converge to $u^{(f)}$ in $\widetilde{W}^{1,2}_{3/2}(\mathbb{R}_T)$, we deduce from (4.7) that, for any $s \in [0,T]$,

$$
\begin{aligned}
1 - \|u^{(f)}(s,\cdot)\|^2_{L^2(\alpha_0)} &= 2\int_s^T \mathrm{d}t \int_{\mathbb{R}} u^{(f)} u_t^{(f)} \mathrm{d}\alpha_0 \\
&= -2\int_s^T \mathrm{d}t \int_{\mathbb{R}} u^{(f)}(\mathscr{L}u^{(f)} - f u^{(f)})\mathrm{d}\alpha_0 \\
&\geq -2\int_s^T \lambda_f(t)\|u^{(f)}(t,\cdot)\|^2_{L^2(\alpha_0)}\mathrm{d}t
\end{aligned}
\tag{4.15}
$$

where

$$
\lambda_f(t) = \sup_{\substack{\phi\in C_c^\infty(\mathbb{R}):\phi\geq 0,\\ \|\phi\|_{L^2(\alpha_0)}=1}} \int_{\mathbb{R}} \phi(\mathscr{L}_t\phi - f(t,\cdot)\phi)\mathrm{d}\alpha_0 .
$$

Recall that $\mathrm{d}\alpha_0 = e^{-\psi}\mathrm{d}x$, so integrating by parts and then taking $\phi \mapsto \sqrt{\phi}$ yields

$$
\begin{aligned}
\lambda_f(t) &= \sup_{\substack{\phi\in C_c^\infty(\mathbb{R}):\phi\geq 0,\\ \|\phi\|_{L^2(\alpha_0)}=1}} \left[-\int_{\mathbb{R}} a(t,\cdot)(\phi')^2\,\mathrm{d}\alpha_0 - \int_{\mathbb{R}} f(t,\cdot)\phi^2\,\mathrm{d}\alpha_0\right] \\
&= \sup_{\substack{\phi\in C_c^\infty(\mathbb{R}):\phi\geq 0,\\ \|\phi\|_{L^1(\alpha_0)}=1}} \left[-\frac{1}{4}\int_{\mathbb{R}} \frac{(\phi')^2}{\phi}\, a(t,\cdot)\mathrm{d}\alpha_0 - \int_{\mathbb{R}} f(t,\cdot)\phi\,\mathrm{d}\alpha_0\right].
\end{aligned}
$$

Further, for any $g \in \widetilde{\mathscr{S}}$ and smooth ϕ such that $\phi\alpha_0 \in M_1(\mathbb{R})$, we get by integration by parts and the Cauchy-Schwarz inequality that

$$
\left|\int_{\mathbb{R}} g_x(t,\cdot)\phi\,\mathrm{d}\alpha_0\right| = \left|\int_{\mathbb{R}} g(t,\cdot)(\phi' - \phi\psi')\mathrm{d}\alpha_0\right| \leq \sqrt{\upsilon_1\upsilon_2} + \|\psi'\|_\infty\sqrt{\upsilon_2},
\tag{4.16}
$$

with $\upsilon_1 = \int_{\mathbb{R}}(\phi')^2/\phi\,\mathrm{d}\alpha_0$ and $\upsilon_2 = \upsilon_2(t) = \int_{\mathbb{R}} g^2(t,\cdot)\phi\,\mathrm{d}\alpha_0$. Hence, for $f = g_x + Cg^2$ with $g \in \widetilde{\mathscr{S}}$, we end up with

$$
\sup_{t\in[0,T]}\{\lambda_f(t)\} \leq \sup_{\upsilon_1,\upsilon_2\geq 0}\left[-\frac{1}{4}\underline{a}\upsilon_1 + \sqrt{\upsilon_1\upsilon_2} + \|\psi'\|_\infty\sqrt{\upsilon_2} - C\upsilon_2\right].
\tag{4.17}
$$

For $C > 0$ large enough, the right side of (4.17) is finite; hence we deduce from (4.15) by applying Grönwall's lemma for $s \mapsto \|u^{(f)}(T-s,\cdot)\|^2_{L^2(\alpha_0)}$ that $\|u^{(f)}(0,\cdot)\|_{L^2(\alpha_0)}$ is bounded uniformly (over such f), as claimed.

Next, let $\widehat{\mathbb{H}}$ denote the closure of $\widetilde{\mathscr{S}}$ with respect to the $L^2(R_x \mathrm{d}m)$-norm. From (4.14) we know that, for some finite C_1, C and all $g \in \widetilde{\mathscr{S}}$,

$$\left| \int_{\mathbb{R}_T} g R_{xx} \, \mathrm{d}m \right| \leq C_1 + C \|g\|^2_{\widehat{\mathbb{H}}}.$$

Hence, $g \mapsto \int_{\mathbb{R}_T} g R_{xx} \, \mathrm{d}m$ extends to a bounded linear functional on $\widehat{\mathbb{H}}$, which by the Riesz representation theorem is of the form $g \mapsto (g, \widehat{h})_{\widehat{\mathbb{H}}}$ for some $\widehat{h} \in \widehat{\mathbb{H}}$. The identity $\int_{\mathbb{R}_T} g R_{xx} \, \mathrm{d}m = \int_{\mathbb{R}_T} g \widehat{h} R_x \, \mathrm{d}m$ for all $g \in \widetilde{\mathscr{S}}$ implies that m-a.e. if $R_x = 0$, then $R_{xx} = 0$, with $\widehat{h} = \frac{R_{xx}}{R_x}$ (subject to our running convention that $\frac{0}{0} = 0$). Consequently,

$$\int_{\mathbb{R}_T} \frac{R_{xx}^2}{R_x} \, \mathrm{d}m = \int_{\mathbb{R}_T} \widehat{h}^2 R_x \, \mathrm{d}m = \|\widehat{h}\|^2_{\widehat{\mathbb{H}}} < \infty. \tag{4.18}$$

Step 4. Following the derivation of (4.18) out of (4.14), if

$$\sup_{g \in \widetilde{\mathscr{S}}} \int_{\mathbb{R}_T} (2gR_t - g^2 R_x) \mathrm{d}m < \infty, \tag{4.19}$$

then $\int_{\mathbb{R}_T} \frac{R_t^2}{R_x} \, \mathrm{d}m < \infty$. Furthermore, plugging into (4.19) the value of R_t from the PDE (3.12), all the terms of which are in $L^{3/2}(\mathbb{R}_T)$, we find that (4.19) amounts to

$$\sup_{g \in \widetilde{\mathscr{S}}} \int_{\mathbb{R}_T} \big(2g[A'(R)R_x + \widehat{h}A(R) + hA(R)] - g^2\big) R_x \, \mathrm{d}m < \infty \tag{4.20}$$

for $\widehat{h} \in \widehat{\mathbb{H}}$ of (4.18) and $h \in L^2(R_x \, \mathrm{d}m)$ of (2.8). Pointwise optimizing in (4.20) over the value of $g(t,x)$ at each point of $\mathbb{R}_T$ bounds the supremum by the finite $\|A'(R)R_x + \widehat{h}A(R) + hA(R)\|^2_{L^2(R_x \mathrm{d}m)}$, thereby completing the proof. □

Remark 4.1. A crucial step in the proof of Proposition 2.2(C) consists of showing that the continuous $u = w + 1$ solving (4.7), with $w \in W_2^{1,2}(\mathbb{R}_T) \cap W_{3/2}^{1,2}(\mathbb{R}_T)$, is further bounded and bounded away from 0. In doing so we relied on the results of [1], but we note in passing that with some additional work such positivity can be obtained from [28, cor. 4.6], bypassing the need for [1].

5 Proof of Proposition 2.5

Fixing throughout $\iota \in (0, 1]$ and $\mu \in M_1^{(\iota)}(\mathbb{R})$, we start with the *convexity* of the functionals from which $J_{\iota,\mu}(\cdot)$ is composed, followed by its use in establishing convergence results for $\widetilde{J}(\cdot)$.

LEMMA 5.1. *The functionals*

$$J^{(1)}(R) = \int_{\mathbb{R}_T} \frac{R_t^2}{R_x}\,\mathrm{d}m, \quad J^{(2)}(R) = \int_{\mathbb{R}_T} \frac{R_{xx}^2}{R_x}\,\mathrm{d}m, \quad J^{(3)}(R) = \int_{\mathbb{R}_T} R_x^3\,\mathrm{d}m, \tag{5.1}$$

are convex on the set $\mathscr{F} = \mathscr{F}_{3/2}$ *of* (1.7).

PROOF. The convexity of $J^{(3)}(\cdot)$ is an immediate consequence of the convexity of $x \mapsto x^3$ on $[0,\infty)$. Further, recall from Steps 3 and 4 in the proof of Proposition 2.2(C), that on $\mathscr{F}$

$$J^{(1)}(R) = \sup_{g \in \widetilde{\mathscr{S}}} \int_{\mathbb{R}_T} \left(2gR_t - g^2 R_x\right)\mathrm{d}m,$$
$$J^{(2)}(R) = \sup_{g \in \widetilde{\mathscr{S}}} \int_{\mathbb{R}_T} \left(2gR_{xx} - g^2 R_x\right)\mathrm{d}m,$$

so each of these functionals, being a supremum of linear functionals, must therefore be convex. $\square$

LEMMA 5.2. *Suppose* $A(\cdot) \geq \underline{a} > 0$ *with* $b(\cdot)$ *and* $A'(\cdot)$ *continuous and bounded. Let* $R = R^{(\gamma)}$ *for* $\gamma \in \mathscr{C}$ *such that* $J_{\iota,\mu}(\gamma) < \infty$ *and suppose the strictly positive probability densities* $R_x^\epsilon \in C^{1,1}(\mathbb{R}_T)$ *are such that*

$$\limsup_{\epsilon \downarrow 0} J^{(\ell)}(R^\epsilon) \leq J^{(\ell)}(R), \quad \ell = 1,2,3, \tag{5.2}$$

$R^\epsilon \to R$ *uniformly on compacts,* $R_x^\epsilon \to R_x$ *in* $L^p(\mathbb{R}_T)$*,* $p \in [2,3]$ *and* m*-a.e.* $R_t^\epsilon \to R_t$*,* $R_{xx}^\epsilon \to R_{xx}$*. If* $\gamma^\epsilon = R_x^\epsilon\,\mathrm{d}x$ *are such that* $\gamma^\epsilon(0) \in M_1^{(0)}(\mathbb{R})$*, then*

$$\lim_{\epsilon \downarrow 0} \widetilde{J}(\gamma^\epsilon) = J_{\iota,\mu}(\gamma). \tag{5.3}$$

PROOF.

Step 1. We first show that as $\epsilon \downarrow 0$:

$$\int_{\mathbb{R}_T} \left| \frac{R_t^\epsilon}{\left(R_x^\epsilon\right)^{1/2}} - \frac{R_t}{(R_x)^{1/2}} \right|^2 \mathrm{d}m \to 0, \tag{5.4}$$

$$\int_{\mathbb{R}_T} \left| \frac{R_{xx}^\epsilon}{\left(R_x^\epsilon\right)^{1/2}} - \frac{R_{xx}}{(R_x)^{1/2}} \right|^2 \mathrm{d}m \to 0, \tag{5.5}$$

$$\int_{\mathbb{R}_T} \left| \left(R_x^\epsilon\right)^{3/2} - (R_x)^{3/2} \right|^2 \mathrm{d}m \to 0. \tag{5.6}$$

To this end, with $|x_1 - x_2|^2 \le |x_1^2 - x_2^2|$ whenever $x_1, x_2 \in \mathbb{R}_+$ and $R_x^\epsilon \to R_x$ in $L^2(\mathbb{R}_T)$, it follows that, for $\epsilon \downarrow 0$,

$$\int_{\mathbb{R}_T} \left|\left(R_x^\epsilon\right)^{1/2} - (R_x)^{1/2}\right|^2 \mathrm{d}m \to 0. \tag{5.7}$$

Similarly, combining the inequality $|x_1^3 - x_2^3| \le \frac{3}{2}|x_1^2 - x_2^2| \max(x_1, x_2)$ (which holds for all $x_1, x_2 \in \mathbb{R}_+$) with Hölder's inequality, we find that

$$\begin{aligned}
&\int_{\mathbb{R}_T} \left|\left(R_x^\epsilon\right)^{3/2} - (R_x)^{3/2}\right|^2 \mathrm{d}m \\
&\quad \le \frac{9}{4} \int_{\mathbb{R}_T} \left|R_x^\epsilon - R_x\right|^2 \max\left(R_x^\epsilon, R_x\right) \mathrm{d}m \\
&\quad \le \frac{9}{4} \left(\int_{\mathbb{R}_T} \left|R_x^\epsilon - R_x\right|^3 \mathrm{d}m\right)^{2/3} \left(\int_{\mathbb{R}_T} \max\left(R_x^\epsilon, R_x\right)^3 \mathrm{d}m\right)^{1/3}.
\end{aligned}$$

By assumption $R_x^\epsilon \to R_x$ in $L^3(\mathbb{R}_T)$ and, due to (5.2) for $\ell = 3$, the norms $\|R_x^\epsilon\|_{L^3(\mathbb{R}_T)}$ are uniformly bounded, thereby yielding (5.6).

Turning to prove (5.4), we let $c^\epsilon = R_t^\epsilon (R_x^\epsilon)^{-1/2}$, which by (5.2) with $\ell = 1$ is $L^2(\mathbb{R}_T)$-bounded. Hence, for any sequence $\epsilon_k \downarrow 0$ we have that $c^\epsilon \to c^*$ weakly in $L^2(\mathbb{R}_T)$ along some subsequence (by the Banach-Alaoglu theorem, where $c^* \in L^2(\mathbb{R}_T)$ may depend on the chosen subsequence). From the triangle inequality, we thus get from (5.7) that along this subsequence, for any fixed $\psi \in C_c(\mathbb{R}_T)$,

$$\int_{\mathbb{R}_T} \left(R_x^\epsilon\right)^{1/2} c^\epsilon \psi \, \mathrm{d}m \to \int_{\mathbb{R}_T} (R_x)^{1/2} c^* \psi \, \mathrm{d}m. \tag{5.8}$$

Since $R \in \mathscr{F}$, we have that $R_t \in L^{3/2}(\mathbb{R}_T)$ and, by the assumption of the lemma, $(R_x^\epsilon)^{1/2} c^\epsilon = R_t^\epsilon \to R_t$ m-a.e. Thus, for any fixed $\psi \in C_c(\mathbb{R}_T)$, the left-hand side of (5.8) converges to $\int_{\mathbb{R}_T} R_t \psi \, \mathrm{d}m$ as $\epsilon \downarrow 0$, resulting in

$$\int_{\mathbb{R}_T} (R_x)^{1/2} c^* \psi \, \mathrm{d}m = \int_{\mathbb{R}_T} R_t \psi \, \mathrm{d}m. \tag{5.9}$$

Furthermore, with $(R_x)^{1/2} \in L^6(\mathbb{R}_T)$ and $c^* \in L^2(\mathbb{R}_T)$, by Hölder's inequality $(R_x)^{1/2} c^* \in L^{3/2}(\mathbb{R}_T)$, so from (5.9) we conclude that m-a.e. $c^* = R_t (R_x)^{-1/2}$, independently of the sequence ϵ_k. That is, $R_t^\epsilon (R_x^\epsilon)^{-1/2} \to R_t (R_x)^{-1/2}$ weakly in $L^2(\mathbb{R}_T)$ when $\epsilon \downarrow 0$. Together with the $L^2(\mathbb{R}_T)$-norm bound of (5.2) for $\ell = 1$, this yields the (strong) convergence of (5.4).

Finally, the same argument, just with R_t^ϵ replaced by R_{xx}^ϵ, yields (5.5).

Step 2. To deduce (5.3) from (5.4)–(5.7), recall that $J_{\iota,\mu}(\gamma) < \infty$ requires $\gamma \in \mathscr{A}_{\iota,\mu}$ and, in particular, that $R = R^{(\gamma)} \in C_b(\mathbb{R}_T)$ satisfies (3.1) for some $M = M_\alpha$

finite (and all $\alpha > 0$). Thus, our assumption that $R^\epsilon \to R$ uniformly on compact sets, combined with the monotonicity of the distribution functions $x \mapsto R^\epsilon(t,x)$, $x \mapsto R(t,x)$, and (3.1), yields that $R^\epsilon \to R$ uniformly on $\mathbb{R}_T$ when $\epsilon \downarrow 0$. This, and the assumed continuity of A' and b on $[0,1]$, show that as $\epsilon \downarrow 0$,

$$\sigma(R^\epsilon) \to \sigma(R), \quad A'(R^\epsilon) \to A'(R), \quad b(R^\epsilon) \to b(R) \quad \text{uniformly on } \mathbb{R}_T. \tag{5.10}$$

Moreover, all functions appearing in (5.10) are uniformly bounded on $\mathbb{R}_T$. Putting this together with the uniform positivity of σ and (5.4)–(5.7), we have the following convergences in $L^2(\mathbb{R}_T)$ when $\epsilon \downarrow 0$:

$$\frac{1}{\sigma(R^\epsilon)} \frac{R_t^\epsilon}{(R_x^\epsilon)^{1/2}} \to \frac{1}{\sigma(R)} \frac{R_t}{(R_x)^{1/2}}, \tag{5.11}$$

$$\sigma(R^\epsilon)\frac{R_{xx}^\epsilon}{(R_x^\epsilon)^{1/2}} \to \sigma(R)\frac{R_{xx}}{(R_x)^{1/2}}, \tag{5.12}$$

$$\frac{A'(R^\epsilon)}{\sigma(R^\epsilon)}\big(R_x^\epsilon\big)^{3/2} \to \frac{A'(R)}{\sigma(R)}(R_x)^{3/2}, \tag{5.13}$$

$$\frac{b(R^\epsilon)}{\sigma(R^\epsilon)}\big(R_x^\epsilon\big)^{1/2} \to \frac{b(R)}{\sigma(R)}(R_x)^{1/2}. \tag{5.14}$$

By Hölder's inequality the finiteness of $J^{(\ell)}(R^\epsilon)$, $\ell = 1,2,3$, implies that $R^\epsilon = R^{(\gamma^\epsilon)} \in \mathscr{F}$, and by our assumptions, also $\gamma^\epsilon \in \widetilde{\mathscr{A}}$. Thus, as $\epsilon \downarrow 0$ we have that

$$\begin{aligned}\widetilde{J}(\gamma^\epsilon) &= \frac{1}{2}\left\| \frac{R_t^\epsilon - (A(R^\epsilon)R_x^\epsilon)_x}{\sigma(R^\epsilon)(R_x^\epsilon)^{1/2}} + \frac{b(R^\epsilon)}{\sigma(R^\epsilon)}\big(R_x^\epsilon\big)^{1/2}\right\|_{L^2(\mathbb{R}_T)}^2 \\ &\longrightarrow \frac{1}{2}\left\| \frac{R_t - (A(R)R_x)_x}{\sigma(R)(R_x)^{1/2}} + \frac{b(R)}{\sigma(R)}(R_x)^{1/2}\right\|_{L^2(\mathbb{R}_T)}^2 = J_{\iota,\mu}(\gamma).\end{aligned}$$

This completes the proof of Lemma 5.2. □

Under Assumption 1.2(a), (b), and (d), our key result provides the conclusions of Proposition 2.5 for $J_{\iota,\mu}(\gamma) < \infty$ and some γ^ϵ in the collection $\mathscr{G}_\iota^\star$ that we define next (cf. Definition 2.4 of $\mathscr{G}$).

DEFINITION 5.3. Let $\mathscr{G}_\iota^\star$ be the subset of $\{\gamma \in \mathscr{C} : J_{\iota,\mu}(\gamma) < \infty, \text{ some } \mu \in M_1^{(\iota)}(\mathbb{R})\}$, such that $R = R^{(\gamma)} \in C_b^\infty(\mathbb{R}_T)$ with R_x strictly positive and

$$\big|\partial_t^j \partial_x^k R(t,x)\big| \le c_{j,k} R_\star(x) \quad \forall (t,x) \in \mathbb{R}_T \tag{5.15}$$

for some $\{c_{j,k},\ \text{finite } j,k \in \mathbb{N}\}$ and

$$R_\star(x) := \sup_{t\in[0,T]}\{\widetilde{R}(t,x)\}, \quad \widetilde{R}(t,x) := 1 - R(t,|x|) + R(t,-|x|). \tag{5.16}$$

PROPOSITION 5.4. *Suppose $A(\cdot) \ge \underline{a} > 0$ with $b(\cdot)$ and $A'(\cdot)$ continuous and bounded. If $J_{\iota,\mu}(\gamma) < \infty$ for some $\mu \in M_1^{(\iota)}(\mathbb{R})$, then there exist $\{\gamma^\epsilon\} \subset \mathscr{G}_\iota^\star$ such that $\gamma^\epsilon \to \gamma$ in $\mathscr{C}$ as $\epsilon \downarrow 0$, $\sup_\epsilon \int |x|^{1+\iota}\,\mathrm{d}\gamma^\epsilon(0)$ is finite, and* (5.3) *holds.*

PROOF. The proof consists of three steps. In Step 1 we construct $\gamma^{\delta,\epsilon} \in \mathscr{C}$ whose smooth CDF paths $R^{\delta,\epsilon} = R^{(\gamma^{\delta,\epsilon})} \in C_b^\infty(\mathbb{R}_T)$ satisfy (5.15) and have strictly positive PDFs $R_x^{\delta,\epsilon}$. Step 2 confirms that $\gamma^{\delta,\epsilon} \to \gamma$ in $\mathscr{C}$ when $(\delta,\epsilon) \to (0,0)$ and that $\gamma^{\delta,\epsilon} \in \mathscr{A}_{\iota,\mu}$ for $\mu = \gamma^{\delta,\epsilon}(0)$ whose $(1+\iota)^{\text{th}}$ moments are bounded, uniformly over (δ,ϵ). Then, relying on Lemma 5.2, we verify in Step 3 that (5.3) holds for $\delta = \delta(\epsilon)$ small enough (so, in particular, such $\gamma^{\delta,\epsilon} \in \mathscr{G}_\iota^\star$).

Step 1. Let $\phi \in C^\infty(\mathbb{R})$ be a strictly positive probability density with

$$\int |x|^k \phi(x)\mathrm{d}x < \infty \quad \text{and} \quad |\phi^{(k)}(x)| \le c_k \phi(x)$$

for some finite c_k, $k \ge 1$, and all $x \in \mathbb{R}$ (for example, the smoothing near $x = 0$ of $e^{-2|x|}$ provides such ϕ). With $\phi_\epsilon(y) = \epsilon^{-1}\phi(y/\epsilon)$, for each $\delta,\epsilon \in (0,1)$ we consider the function

$$S^{\delta,\epsilon}(t,x) = \int_{\mathbb{R}} R((t-3\delta)_+, y)\phi_\epsilon(x-y)\mathrm{d}y \tag{5.17}$$

on $\mathbb{R}_{T+3\delta}$. Then, fixing a probability density $\psi \in C_c^\infty(\mathbb{R})$ supported on $[0,3]$ with

$$c_\psi = \inf_{s\in[1,2]}\{\psi(s)\} > 0, \tag{5.18}$$

we set $\psi_\delta(s) = \delta^{-1}\psi(s/\delta)$ and consider

$$R^{\delta,\epsilon}(t,x) = \int_0^{T+3\delta} S^{\delta,\epsilon}(s,x)\psi_\delta(s-t)\mathrm{d}s, \quad \delta,\epsilon \in (0,1),\ (t,x) \in \mathbb{R}_T. \tag{5.19}$$

With $\psi(t)\phi(x)$ a probability density on $\mathbb{R}_T$, since each $R(t,\cdot)$ is a CDF, so are $S^{\delta,\epsilon}(t,\cdot)$ and $R^{\delta,\epsilon}(t,\cdot)$. By the strict positivity of ϕ we have the same for $S_x^{\delta,\epsilon}$, and thereby also for $R_x^{\delta,\epsilon}$. Next, with $\phi(\cdot)$ smooth, $S^{\delta,\epsilon}(t,\cdot) \in C_b^\infty(\mathbb{R})$ for each t, ϵ, and δ, hence $R^{\delta,\epsilon} \in C_b^\infty(\mathbb{R}_T)$ by the smoothness of ψ. Moreover, as $\phi(\cdot)$ pointwise controls its derivatives, it follows that, for each $k \ge 1$ and all $(t,x) \in \mathbb{R}_T$,

$$\left|\partial_x^k S^{\delta,\epsilon}(t,x)\right| \le c_k \epsilon^{-k} S^{\delta,\epsilon}(t,x).$$

Furthermore, with $\int \phi_\epsilon(x-y)\mathrm{d}y = 1$, the same bound holds for $1-S^{\delta,\epsilon}$, resulting with

$$\left|\partial_x^k S^{\delta,\epsilon}(t,x)\right| \le c_k \epsilon^{-k}[(1-S^{\delta,\epsilon}(t,x)) \wedge S^{\delta,\epsilon}(t,x)] \le c_k \epsilon^{-k} \widetilde{S}^{\delta,\epsilon}(t,x) \tag{5.20}$$

(where $\widetilde{S}^{\delta,\epsilon}$ is related to $S^{\delta,\epsilon}$ analogously to (5.16)). Thus, having $\psi(\cdot)$ smooth yields, by (5.20) and (5.19), that

$$\left|\partial_t^j \partial_x^k R^{\delta,\epsilon}(t,x)\right| \le \left\|\psi_\delta^{(j)}\right\|_\infty c_k \epsilon^{-k} V^{\delta,\epsilon}(t,x) \tag{5.21}$$

for all $j,k \in \mathbb{N}$ where

$$V^{\delta,\epsilon}(t,x) = \int_0^{3\delta} \widetilde{S}^{\delta,\epsilon}(t+u,x)\,\mathrm{d}u.$$

Clearly, for any $\delta > 0$ and $(t, x) \in \mathbb{R}_T$,

$$V^{\delta,\epsilon}(t,x) \le \frac{3\delta}{c_\psi} \sup_{t\in[0,T]} \left\{ \int_0^{3\delta} \widetilde{S}^{\delta,\epsilon}(t+u,x)\psi_\delta(u)\mathrm{d}u \right\} = \frac{3\delta}{c_\psi} R_\star^{\delta,\epsilon}(x)$$

for $R_\star(\cdot)$ of (5.16), which together with (5.21) implies that $R^{\delta,\epsilon}$ satisfies (5.15).

Step 2. Since $S^{\delta,\epsilon}(t,x) = S^{\delta,\epsilon}(0,x)$ when $t \in [0, 3\delta]$, upon specializing (5.19) to $t = 0$, we get the smooth CDFs

$$R^{\delta,\epsilon}(0,x) = \mathbb{E}[R(0, x - Y_\epsilon)] =: \Theta_\epsilon(x) \tag{5.22}$$

for Y_ϵ of density ϕ_ϵ. Our assumption that $\gamma(0) = \mu \in M_1^{(\iota)}(\mathbb{R})$ translates to $\gamma^{\delta,\epsilon}(0) \in M_1^{(\iota)}(\mathbb{R})$ with uniformly bounded $(1+\iota)$-moments (since Y_ϵ has this property and $f \mapsto (\int f(x)^{q_0}\,\mathrm{d}x, \int |x|^{1+\iota} f(x)\,\mathrm{d}x)$ is convex over probability densities). Similarly, from (5.17) and (5.19)

$$\begin{aligned}\int_{\mathbb{R}_T} |x|^{1+\iota}\,\mathrm{d}\gamma^{\delta,\epsilon}(t)\mathrm{d}t &\le \int_{\mathbb{R}_T} \mathbb{E}[|x - Y_\epsilon|^{1+\iota}]\mathrm{d}\gamma(t)\mathrm{d}t \\ &\quad + 3\delta \int_{\mathbb{R}} \mathbb{E}[|x - Y_\epsilon|^{1+\iota}]\mathrm{d}\gamma(0),\end{aligned}$$

with the right-hand side finite, since γ satisfies (1.6), $\gamma(0) \in M_1^{(\iota)}(\mathbb{R})$, and Y_ϵ has finite moments of all orders. That is, $\gamma^{\delta,\epsilon}$ satisfy the moment condition (1.6) and are thus in $\mathscr{A}_{\iota,\gamma^{\delta,\epsilon}(0)}$. In addition, by dominated convergence, $R^{\delta,\epsilon} \to R$ uniformly on compacts, so in particular $\gamma^{\delta,\epsilon} \to \gamma$ in $\mathscr{C}$.

Step 3. By construction, m-a.e. $R_x^{\delta,\epsilon} \to R_x$, $R_t^{\delta,\epsilon} \to R_t$, and $R_{xx}^{\delta,\epsilon} \to R_{xx}$ whenever $(\delta,\epsilon) \to (0,0)$. Thus, from Lemma 5.2 we get (5.3) for $\delta = \delta(\epsilon)$ small, once we show that

$$\limsup_{\epsilon\downarrow 0} \limsup_{\delta\downarrow 0} J^{(\ell)}(R^{\delta,\epsilon}) \le J^{(\ell)}(R), \quad \ell = 1,2,3 \tag{5.23}$$

(indeed, by Egorov's theorem, (5.23) for $\ell = 3$ implies that $R_x^{\delta,\epsilon} \to R_x$ in $L^p(\mathbb{R}_T)$, $p \in [2,3]$). Turning to prove (5.23), we recall from its definition that $J_{\iota,\mu}(\gamma) < \infty$ implies $R = R^{(\gamma)} \in \mathscr{F}$ and

$$J^{(\ell)}(R) = \int_0^{3\delta} \psi_\delta(s)\mathrm{d}s \int_{\mathbb{R}} J^{(\ell)}(R(\cdot, y + \cdot))\phi_\epsilon(y)\mathrm{d}y < \infty, \quad \ell = 1,2,3.$$

Hence, starting with the functional $J^{(1)}(\cdot)$, we get from the definition of $R^{\delta,\epsilon}$, upon applying Lemma 5.1 twice, that

$$J^{(1)}(R^{\delta,\epsilon}) \le \int_0^{3\delta} J^{(1)}(S^{\delta,\epsilon}(s+\cdot,\cdot))\psi_\delta(s)\mathrm{d}s \le J^{(1)}(R)$$

and consequently (5.23) holds for $\ell = 1$. Next, consider the functionals

$$\widehat{J}^{(2)}(F) = \int_{\mathbb{R}} \frac{F''(x)^2}{F'(x)}\,\mathrm{d}x, \quad \widehat{J}^{(3)}(F) = \int_{\mathbb{R}} F'(x)^3\,\mathrm{d}x, \tag{5.24}$$

which, as in the proof of Lemma 5.1, are convex on the set of twice differentiable CDFs. Thus, for $\ell = 2, 3$, we get by the same argument as before that

$$J^{(\ell)}(R^{\delta,\epsilon}) \le J^{(\ell)}(R) + 3\delta \widehat{J}^{(\ell)}(\Theta_\epsilon).$$

From our choice of ϕ_ϵ, we have $|\Theta''_\epsilon| \le c_1\epsilon^{-1}\Theta'_\epsilon$ and uniformly bounded PDF Θ'_ϵ. Consequently, $\widehat{J}^{(\ell)}(\Theta_\epsilon) < \infty$ and taking $\delta \downarrow 0$ establishes (5.23) for $\ell = 2, 3$. $\square$

Specializing to $\iota = 1$ and applying Proposition 5.4 to approximate the given $\gamma \in \mathscr{C}$ of finite $J_{1,\mu}(\gamma)$ by a suitable sequence from $\mathscr{G}_1^\star$, the proof of Proposition 2.5 is thus completed by considering our next lemma (to get $\gamma^\ell \in \mathscr{G}$ that suitably converge to any given $\gamma \in \mathscr{G}_1^\star$).

LEMMA 5.5. *Suppose $\gamma \in \mathscr{G}_\iota^\star$ and Assumption* 1.2(a), (b) *and* (d) *hold. Then, there exist $\mu \in M_1^{(\iota-1)}(\mathbb{R})$ such that $\gamma^\ell = (1-\ell^{-1})\gamma + \ell^{-1}\mu \in \mathscr{G}$ with $\widetilde{J}(\gamma^\ell) \to \widetilde{J}(\gamma)$ and $\gamma^\ell \to \gamma$ in $\mathscr{C}$.*

PROOF. Recall $c_{0,1} = c_{0,1}(\gamma)$ from (5.15). We begin by showing that, with $\eta = 1/(12c_{0,1})$, the function $R_\star$ of (5.16) is such that

$$|x - y| \le 3\eta \quad \Longrightarrow \quad R_\star(y) \le 2R_\star(x). \tag{5.25}$$

Indeed, since $R_\star(z) = R_\star(-z)$ is continuous and nonincreasing in $|z|$, it suffices to prove that $R_\star(y) \le 2R_\star(x)$ for $|x| \in [|y|, |y| + 3\eta]$ and $xy \ge 0$. To this end, note that by (5.15), for all $t \in [0, T]$,

$$|R(t, y) - R(t, x)| = \left|\int_x^y R_x(t, z)\mathrm{d}z\right| \le c_{0,1}\left|\int_x^y R_\star(z)\mathrm{d}z\right| \le \frac{1}{4}R_\star(y).$$

The preceding implies that, for $\widetilde{R}(t, x)$ of (5.16),

$$\widetilde{R}(t, y) \le \widetilde{R}(t, x) + \frac{1}{2}R_\star(y),$$

so taking the maximum over t results in $R_\star(y) \le R_\star(x) + \frac{1}{2}R_\star(y)$, as claimed.

We next claim that

$$\int_{\mathbb{R}} |x|^\iota R_\star(x)\mathrm{d}x < \infty. \tag{5.26}$$

Indeed, taking $\widehat{\eta} = 1/(8c_{1,0}) > 0$, we have that, for $|t - s| \le 2\widehat{\eta}$,

$$|R(t, x) - R(s, x)| \le c_{1,0}|t - s|R_\star(x) \le \frac{1}{4}R_\star(x),$$

out of which we deduce that $\widetilde{R}(s,x) \le \widetilde{R}(t,x) + \frac{1}{2} R_\star(x)$. Considering $s \in [0,T]$ such that $\widetilde{R}(s,x) = R_\star(x)$, we thus find that $\widetilde{R}(t,x) \ge \frac{1}{2} R_\star(x)$ throughout a subinterval of length at least $2\widehat{\eta}$. Consequently, for all $x \ge 0$,

$$\widehat{\eta} R_\star(x) \le \int_0^T \widetilde{R}(t,x)\mathrm{d}t,$$

hence by Fubini's theorem,

$$\frac{\iota+1}{2}\widehat{\eta} \int_{\mathbb{R}} |x|^\iota R_\star(x)\mathrm{d}x \le (\iota+1)\int_0^\infty x^\iota \,\mathrm{d}x \int_0^T \widetilde{R}(t,x)\mathrm{d}t = \int_{\mathbb{R}_T} |x|^{\iota+1}\,\mathrm{d}\gamma(t)\mathrm{d}t,$$

which for $\gamma \in \mathscr{G}_\iota^\star$ is finite in view of (1.6).

Continuing with the proof of the lemma, let $\kappa = \int_{\mathbb{R}} R_\star(y)\mathrm{d}y$, which by (5.26) is finite. Then, for $\eta > 0$ of (5.25) and $\psi \in C_c^\infty(\mathbb{R})$ supported on $[0,3]$ as in Proposition 5.4, we construct the PDF

$$r'(x) := \frac{1}{\kappa}\int_{\mathbb{R}} R_\star(y)\psi_\eta(x-y)\mathrm{d}y, \tag{5.27}$$

and, for each $\epsilon > 0$, consider the path $\gamma^\epsilon \in \mathscr{C}$ associated with the CDFs

$$R^\epsilon(t,x) := (1-\epsilon)R(t,x) + \epsilon r(x).$$

Since R_x is strictly positive, so are R_x^ϵ. Furthermore, $r \in C_b^\infty(\mathbb{R})$, and consequently also $R^\epsilon \in C_b^\infty(\mathbb{R}_T)$. For $\epsilon \downarrow 0$ we clearly have that $R^\epsilon \to R$ uniformly on compacts (so $\gamma^\epsilon \to \gamma$ in $\mathscr{C}$), and m-a.e. $R_t^\epsilon \to R_t$, $R_{xx}^\epsilon \to R_{xx}$. Since $r'(\cdot)$ and $R_x(t,\cdot)$ are both uniformly bounded PDFs, obviously also $R_x^\epsilon \to R_x$ in $L^p(\mathbb{R}_T)$ for all $p \in [2,3]$. Thus, in view of Lemmas 5.1 and 5.2, we have that $\widetilde{J}(\gamma^\epsilon) \to \widetilde{J}(\gamma)$ provided we show that $\widehat{J}^{(\ell)}(r)$, $\ell = 2,3$, of (5.24) are finite. The finiteness of $\widehat{J}^{(3)}(r)$ is trivial, for r' is a bounded PDF, whereas $\widehat{J}^{(2)}(r) < \infty$ due to the boundedness of $|r''(x)|/r'(x)$. To see the latter, note that for $c_\psi > 0$ as in (5.18), by (5.25) and (5.27),

$$\begin{aligned} \frac{c_\psi}{2\eta} R_\star(x) \le \frac{c_\psi}{\eta} \inf_{y\in[x-3\eta,x]}\{R_\star(y)\} &\le \kappa r'(x) \\ &\le \sup_{y\in[x-3\eta,x]}\{R_\star(y)\} \le 2R_\star(x). \end{aligned} \tag{5.28}$$

Similarly, $\kappa|r''(x)| \le 2\|\psi'\|_1 \eta^{-1} R_\star(x)$, which, together with the left-hand side of (5.28), implies that $|r''(x)|/r'(x)$ is uniformly bounded (by $4\|\psi'\|_1/c_\psi$).

Next, combining the right-hand side of (5.28) with (5.26), we deduce that the ι^{th} moment of $\mu = r'(x)\mathrm{d}x$ is finite. It thus remains only to show that the continuous function

$$h(R^\epsilon) = \frac{R_t^\epsilon - (A(R^\epsilon)R_x^\epsilon)_x}{A(R^\epsilon)R_x^\epsilon}$$

is further uniformly bounded and globally Lipschitz in x. With A bounded below, A', R_x^ϵ bounded above, and $R_t^\epsilon = (1-\epsilon)R_t$, the boundedness of $h(R^\epsilon)$ follows from that of $(|R_t|+|R_{xx}|+|r''|)/r'$. To this end, we have just shown the uniform boundedness of $|r''(x)|/r'(x)$ and recall from (5.15) that $|R_t(t,x)|+|R_{xx}(t,x)| \le (c_{1,0}+c_{0,2})R_\star(x)$, which by the left-hand side of (5.28) is further bounded by $Cr'(x)$ for some $C = C(\gamma)$ finite. As for showing that $h(R^\epsilon)$ is globally Lipschitz continuous in x, note that

$$[h(R^\epsilon)]_x = \frac{(1-\epsilon)R_{tx} - (A(R^\epsilon)R^\epsilon_x)_{xx} - h(R^\epsilon)(A(R^\epsilon)R^\epsilon_x)_x}{A(R^\epsilon)R^\epsilon_x}.$$

Recall that $A(\cdot)$ is bounded below, $h(R^\epsilon)$, R_x, r', and $(|R_{xx}|+|r''|)/r'$ are bounded above, and A' is globally Lipschitz. We thus have that $|[h(R^\epsilon)]_x|$ is uniformly bounded, provided that $(|R_{tx}|+|R_{xxx}|+|r'''|)/r'$ is uniformly bounded. The latter holds by the left-hand side of (5.28), since from (5.20) we have that $|R_{tx}|+|R_{xxx}| \le CR_\star$ for some $C = C(\gamma)$ finite, and by the same reasoning as above, we also get that $\kappa|r'''| \le 2\|\psi''\|_1\eta^{-2}R_\star$. □

6 Proof of Proposition 2.6

We suppose throughout this section that parts (a)–(c) of Assumption 1.2 hold for some $\iota_\star \in (0,1]$ and use the simplified notations $I(\cdot)$ and $\mathscr{A}$ for $I_{\iota_\star,\rho_0}(\cdot)$ of (2.4) and $\mathscr{A}_{\iota_\star,\rho_0}$ of (1.6), respectively. In this setting we establish a local large deviations upper bound for ρ^N being near γ, starting with $\gamma \in \mathscr{A}$ (where $I(\gamma) = \sup_{g\in\widetilde{\mathscr{S}}}[\Phi_\gamma(g) - (g,g)_\gamma]$).

PROPOSITION 6.1. *For each $\gamma \in \mathscr{A}$ and any $g \in \widetilde{\mathscr{S}}$ we have the upper bound*

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(\rho^N \in B(\gamma,\delta)) \le (g,g)_\gamma - \Phi_\gamma(g). \tag{6.1}$$

PROOF. Fixing $g \in \widetilde{\mathscr{S}}$, for each $t\in[0,T]$ and $\xi \in \mathscr{C}$, we set

$$H^g(\xi)(t) = (\xi, \mathscr{R}^\xi g)(t),$$

with $\mathscr{R}^\xi g = g_t + b(R^{(\xi)})g_x + A(R^{(\xi)})g_{xx}$ of (2.1). Then, applying Itô's formula for the real-valued stochastic processes $Z^g_N(t) := (\rho^N, g)(t)$, one finds that

$$Z^g_N(t) - Z^g_N(0) = \int_0^t H^g(\rho^N)(s)\mathrm{d}s + M^g_N(t), \tag{6.2}$$

with the continuous martingale

$$M^g_N(t) := \frac{1}{N}\sum_{i=1}^N\int_0^t \sigma\big(F_{\rho^N(s)}(X_i(s))\big)g_x(s,X_i(s))\mathrm{d}W_i(s). \tag{6.3}$$

Its quadratic variation is $\langle M^g_N\rangle(t) = \frac{1}{N}\int_0^t V^g(\rho^N)(s)\mathrm{d}s$, with

$$V^g(\xi)(t) := 2\big(\xi, A(R^{(\xi)})g_x^2\big)(t). \tag{6.4}$$

Hence, by the martingale representation theorem (see [24, theorem 3.4.2]),

$$M_N^g(t) = \frac{1}{\sqrt{N}} \int_0^t \sqrt{V^g(\rho^N)(s)}\, \mathrm{d}\beta_N(s)$$

for some one-dimensional standard Brownian motion β_N. Next, fixing $\gamma \in \mathscr{A}$, let

$$\overline{H}^g(\gamma)(t) = \int_0^t H^g(\gamma)(s)\mathrm{d}s, \quad \overline{V}^g(\gamma)(t) = \int_0^t V^g(\gamma)(s)\mathrm{d}s, \tag{6.5}$$

and recall that, by (2.2) and (2.3),

$$(g,g)_\gamma - \Phi_\gamma(g) = \frac{1}{2}\overline{V}^g(\gamma)(T) + (\gamma, g)(0) + \overline{H}^g(\gamma)(T) - (\gamma, g)(T). \tag{6.6}$$

We thus proceed to compare $Z_N^g(\cdot)$ with the process

$$Y_N^g(t) = (\gamma, g)(0) + \overline{H}^g(\gamma)(t) + M_N^{g,\gamma}(t), \quad t \in [0, T], \tag{6.7}$$

having the martingale part

$$M_N^{g,\gamma}(t) = \frac{1}{\sqrt{N}} \int_0^t \sqrt{V^g(\gamma)(s)}\, \mathrm{d}\beta_N(s), \quad t \in [0, T].$$

Step 1. We first show that, on the event $\rho^N \in B(\gamma, \delta)$,

$$\left\| Z_N^g - Y_N^g \right\|_\infty := \sup_{t\in[0,T]} \left| Z_N^g(t) - Y_N^g(t) \right| \le \epsilon,$$

up to a probability that is negligible at our large deviations exponential scale (in the limit $N \to \infty$ followed by $\delta \downarrow 0$). To this end, fixing $\rho \in B(\gamma, \delta)$ we note that

$$\begin{aligned}\int_0^T |H^g(\rho)(s) - H^g(\gamma)(s)|\mathrm{d}s \le &\int_0^T |(\rho, \mathscr{R}^\rho g)(s) - (\rho, \mathscr{R}^\gamma g)(s)|\mathrm{d}s \\ &+ \int_0^T |(\rho, \mathscr{R}^\gamma g)(s) - (\gamma, \mathscr{R}^\gamma g)(s)|\mathrm{d}s,\end{aligned}$$

and setting $\widetilde{R} = R^{(\rho)}$, $R = R^{(\gamma)}$, we further have

$$\begin{aligned}\int_0^T |V^g(\rho)(s) - V^g(\gamma)(s)|\mathrm{d}s \le 2&\int_0^T \left|\big(\rho, A(\widetilde{R})g_x^2\big)(s) - \big(\rho, A(R)g_x^2\big)(s)\right|\mathrm{d}s \\ &+ 2\int_0^T \left|\big(\rho, A(R)g_x^2\big)(s) - \big(\gamma, A(R)g_x^2\big)(s)\right|\mathrm{d}s.\end{aligned}$$

Since $\gamma \in \mathscr{A}$, we know that $R \in C_b(\mathbb{R}_T)$ and consequently so are $\mathscr{R}^\gamma g$ and $A(R)g_x^2$, from which we deduce that the second term in both upper bounds tends to 0 as $\delta \downarrow 0$, uniformly in $\rho \in B(\gamma, \delta)$. Now, recall that $b(\cdot)$ and $A(\cdot)$ are Lipschitz functions (under Assumption 1.2(a)–(b)). Hence, to get the same uniform convergence for the first term in both upper bounds, it suffices to show that

$$\lim_{\delta\downarrow 0} \sup_{\rho\in B(\gamma,\delta)} \int_0^T (\rho, |\widetilde{R} - R|)(s)\mathrm{d}s = 0. \tag{6.8}$$

Moreover, fixing $\delta > 0$, by definition of the metric $d(\cdot,\cdot)$ on $\mathscr{C}$, for any $\rho \in B(\gamma,\delta)$ and $(s,x) \in \mathbb{R}_T$ one has that

$$R(s,x-\delta) - \delta \le \widetilde{R}(s,x) \le R(s,x+\delta) + \delta \tag{6.9}$$

(see (1.5)), and hence, also

$$|\widetilde{R}(s,x) - R(s,x)| \le R(s,x+\delta) - R(s,x-\delta) + \delta.$$

Recall that $R(s,x+\delta) - R(s,x-\delta) = \gamma(s,(x-\delta,x+\delta])$; hence by Fubini's theorem and yet another application of the preceding bound

$$\begin{aligned}(\rho, |\widetilde{R} - R|)(s) &\le \delta + \int_{\mathbb{R}} (R(s,x+\delta) - R(s,x-\delta))\rho(s,\mathrm{d}x) \\ &= \delta + \int_{\mathbb{R}} \rho(s,[y-\delta,y+\delta))\gamma(s,\mathrm{d}y) \\ &\le 3\delta + \int_{\mathbb{R}} (R(s,y+3\delta) - R(s,y-3\delta))\gamma(s,\mathrm{d}y).\end{aligned}$$

Integrating both sides over $s \in [0,T]$, we see that (6.8) is a consequence of

$$\lim_{\delta\downarrow 0} \int_{\mathbb{R}_T} (R(s,y+3\delta) - R(s,y-3\delta))\gamma(s,\mathrm{d}y)\mathrm{d}s = 0,$$

which in turn follows from the dominated convergence theorem and the fact that $R \in C_b(\mathbb{R}_T)$. All in all, we have shown that

$$\lim_{\delta\downarrow 0} \sup_{\rho\in B(\gamma,\delta)} \int_0^T |H^g(\rho)(s) - H^g(\gamma)(s)|\mathrm{d}s = 0, \tag{6.10}$$

$$\lim_{\delta\downarrow 0} \sup_{\rho\in B(\gamma,\delta)} \int_0^T |V^g(\rho)(s) - V^g(\gamma)(s)|\mathrm{d}s = 0. \tag{6.11}$$

Recall that $(\rho^N, g)(0) \to (\rho_0, g(0,\cdot)) = (\gamma, g)(0)$ by Assumption 1.2(c) and the definition of $\mathscr{A}$. Hence, comparing (6.2) with (6.7), it immediately follows from (6.10) that, for any fixed $\epsilon > 0$,

$$\begin{aligned}&\lim_{\delta\downarrow 0}\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}\big(\rho^N \in B(\gamma,\delta), \|Z_N^g - Y_N^g\|_\infty > 2\epsilon\big) \le \\ &\qquad \lim_{\delta\downarrow 0}\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}\big(\rho^N \in B(\gamma,\delta), \|M_N^g - M_N^{g,\gamma}\|_\infty > \epsilon\big).\end{aligned}$$

Recall [32, theorem 8.5.7] that $M_N^g(\cdot) - M_N^{g,\gamma}(\cdot)$ has the law of time-changed standard Brownian motion $\beta(\tau(\cdot))$, for $\tau(t) = \langle M_N^g - M_N^{g,\gamma}\rangle(t)$ (the quadratic variation process of the martingale $M_N^g - M_N^{g,\gamma}$). Moreover, on the event $\{\rho^N \in B(\gamma,\delta)\}$, we have that

$$\tau(T) = \frac{1}{N}\int_0^T \left(\sqrt{V^g(\rho^N)(s)} - \sqrt{V^g(\gamma)(s)}\right)^2 \mathrm{d}s \le \frac{1}{N}\kappa_\gamma(\delta),$$

with $\kappa_\gamma(\delta) \to 0$ as $\delta \downarrow 0$ (due to the inequality $(\sqrt{x_1} - \sqrt{x_2})^2 \le |x_1 - x_2|$ for $x_1, x_2 \ge 0$ and (6.11)). Thus, from Bernstein's inequality for Brownian motion,

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\rho^N \in B(\gamma,\delta), \big\|M_N^g - M_N^{g,\gamma}\big\|_\infty > \epsilon\big)\\
&\quad\le \lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\Big(\sup_{t\in[0,\kappa_\gamma(\delta)/N]}|\beta(t)| > \epsilon\Big) \le -\lim_{\delta\downarrow 0}\left\{\frac{\epsilon^2}{2\kappa_\gamma(\delta)}\right\} = -\infty.
\end{aligned}$$

Step 2. Recall that, for any $\alpha_1, \alpha_2 \in M_1(\mathbb{R})$,

$$d_{BL}(\alpha_1,\alpha_2) = \sup_{\|f\|_\infty + \|f\|_{\mathrm{Lip}}\le 1}\{|(\alpha_1, f) - (\alpha_2, f)|\} \le 2 d_L(\alpha_1,\alpha_2) \tag{6.12}$$

(see [13, cor. 11.6.5]). Hence, by (1.5) there exists $r_g : (0,\infty) \to (0,\infty)$ such that $\lim_{\delta\downarrow 0} r_g(\delta) = 0$, and

$$\rho \in B(\gamma,\delta) \quad\Longrightarrow\quad \|(\rho, g) - (\gamma, g)\|_\infty \le r_g(\delta).$$

Recalling that $Z_N^g(t) = (\rho^N, g)(t)$, we thus have from Step 1 that, for any $\epsilon > 0$,

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(\rho^N \in B(\gamma,\delta))\\
&\quad= \lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\rho^N \in B(\gamma,\delta), \big\|Z_N^g - Y_N^g\big\|_\infty \le \epsilon\big)\\
&\quad\le \lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\|(\rho^N, g) - (\gamma, g)\|_\infty \le r_g(\delta), \big\|Z_N^g - Y_N^g\big\|_\infty \le \epsilon\big)\\
&\quad\le \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\big\|(\gamma, g) - Y_N^g\big\|_\infty \le 2\epsilon\big).
\end{aligned}$$

Therefore, it suffices to prove the simpler local large deviations upper bound

$$\lim_{\epsilon\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\big\|(\gamma, g) - Y_N^g\big\|_\infty \le \epsilon\big) \le -I^g((\gamma, g)), \tag{6.13}$$

provided that (see (6.6))

$$I^g((\gamma, g)) \ge (\gamma, g)(T) - (\gamma, g)(0) - \overline{H}^g(\gamma)(T) - \frac{1}{2}\overline{V}^g(\gamma)(T). \tag{6.14}$$

Step 3. We establish (6.13) as a consequence of the LDP holding for $\{Y_N^g\}$ in the space $C([0,T],\mathbb{R})$ with a good rate function $I^g(\cdot)$. Indeed, note that by the time-change formalism for Itô integrals (see, e.g., [32, theorem 8.5.7]), the process Y_N^g

can be obtained as $\Psi(N^{-1/2}\widetilde{\beta})$ for a one-dimensional standard Brownian motion $\widetilde{\beta}(t)$, $t \geq 0$, and the deterministic operator

$$\Psi : C([0,\infty),\mathbb{R}) \to C([0,T],\mathbb{R}),$$
$$(\Psi h)(t) = (\gamma, g)(0) + \overline{H}^g(\gamma)(t) + h(\overline{V}^g(\gamma)(t)).$$

Clearly, Ψ is continuous with respect to uniform convergence on compacts in $C([0,\infty),\mathbb{R})$. Hence, by Schilder's theorem (see [10, theorem 5.2.3]) and the contraction principle (see [10, theorem 4.2.1]), the sequence $\{Y_N^g\}$ satisfies the LDP in $C([0,T],\mathbb{R})$ with the good rate function

$$I^g(f) = \inf_{\{h:\Psi(h)=f\}} \frac{1}{2}\int_0^S \left(\frac{dh}{du}\right)^2 \mathrm{d}u, \tag{6.15}$$

where $S = \overline{V}^g(\gamma)(T)$ and the infimum is over all absolutely continuous functions h on $[0,S]$, starting at $h(0)=0$, with Radon-Nikodym derivative $\frac{dh}{du} \in L^2([0,S])$. In particular, since

$$0 \leq \frac{1}{2}\int_0^S \left(\frac{dh}{du} - 1\right)^2 \mathrm{d}u = \frac{1}{2}\int_0^S \left(\frac{dh}{du}\right)^2 \mathrm{d}u - h(S) + \frac{1}{2}S,$$

it follows from the requirement $(\Psi h)(T) = (\gamma, g)(T)$ that

$$I^g((\gamma,g)) \geq h(S) - \frac{1}{2}S = (\gamma,g)(T) - (\gamma,g)(0) - \overline{H}^g(\gamma)(T) - \frac{1}{2}S,$$

which is precisely our claim (6.14). □

We proceed with the local large deviations upper bound for paths $\gamma \notin \mathscr{A}$.

PROPOSITION 6.2. *If $\gamma \notin \mathscr{A}$, then*

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}(\rho^N \in B(\gamma,\delta)) = -\infty. \tag{6.16}$$

PROOF. Fixing N, let $\mathbb{Q}^{(b)}$ denote the law of the solution of the SDS (1.2) with $\mathbb{Q} = \mathbb{Q}^{(0)}$ corresponding to the solution of (1.2) in the case $b \equiv 0$. Recall that

$$\frac{\mathrm{d}\mathbb{Q}^{(b)}}{\mathrm{d}\mathbb{Q}} = \exp\left(M_N^b(T) - \frac{1}{2}\langle M_N^b\rangle(T)\right) \tag{6.17}$$

(see [24, theorem 3.5.1]), with the continuous martingale

$$M_N^b(t) = \sum_{i=1}^N \int_0^t \frac{b(F_{\rho^N(s)}(X_i(s)))}{\sigma(F_{\rho^N(s)}(X_i(s)))}\,\mathrm{d}W_i(s), \quad t \in [0,T],$$

whose quadratic variation is $\langle M_N^b\rangle(t) = N\int_0^t U_N^2(s)\mathrm{d}s$, with $U_N(s)$ uniformly bounded by $\|b/\sigma\|_\infty$. Hence, setting $\kappa = \frac{T}{2}\|b/\sigma\|_\infty^2$, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{Q}^{(b)}(\rho^N \in B(\gamma,\delta)) &= \mathbb{Q}\big[e^{M_N^b(T)-\frac{1}{2}\langle M_N^b\rangle(T)}\mathbb{1}_{\{\rho^N\in B(\gamma,\delta)\}}\big] \\ &\le e^{\kappa N}\mathbb{Q}\big[e^{2M_N^b(T)-2\langle M_N^b\rangle(T)}\big]^{1/2}\mathbb{Q}\big(\rho^N \in B(\gamma,\delta)\big)^{1/2} \\ &= e^{\kappa N}\,\mathbb{Q}^{(2b)}[1]^{1/2}\mathbb{Q}(\rho^N \in B(\gamma,\delta))^{1/2}. \end{aligned}\tag{6.18}$$

Consequently, it suffices to establish (6.16) when $b \equiv 0$ in order to have the same conclusion for any other choice of $b(\cdot)$. With $\mathscr{A}$ independent of such choice, we proceed throughout the proof with $b \equiv 0$ (without loss of generality). In addition, since $\rho^N \to \rho_0$, the bound (6.16) trivially holds when $\gamma(0) \neq \rho_0$. Assuming hereafter that $\gamma(0) = \rho_0$, we distinguish the three reasons for $\gamma \notin \mathscr{A}$. We start with case (a) where $R^{(\gamma)} \notin C_b(\mathbb{R}_T)$, proceed to case (b) in which $R^{(\gamma)} \in C_b(\mathbb{R}_T)$ but γ fails to satisfy the moment condition (1.6) for $\iota = \iota_\star$ of Assumption 1.2(c), and finish with case (c), where $R^{(\gamma)} \in C_b(\mathbb{R}_T)$ and (1.6) holds, but $t \mapsto (\gamma, g)(t)$ is not absolutely continuous for some $g \in \widetilde{\mathscr{S}}$.

Case (a). By assumption $\gamma \in \mathscr{C}$ and $\gamma(0)$ has a density, so if $R^{(\gamma)} \notin C_b(\mathbb{R}_T)$, then necessarily $\gamma(s)(\{y\}) = 3\epsilon$ for some $s \in (0, T]$, $y \in \mathbb{R}$, and $\epsilon > 0$. Fixing N and $0 < \delta < \epsilon$, it follows from (1.5), the definition of $d_L(\cdot,\cdot)$, and the union bound that, for $m = \lceil N\epsilon \rceil$,

$$\begin{aligned}\mathbb{P}(\rho^N \in B(\gamma,\delta)) &\le \mathbb{P}(\rho^N(s)([y-\delta, y+\delta]) \ge \epsilon) \\ &= \mathbb{P}\bigg(\sum_{i=1}^N \mathbb{1}_{\{|X_i(s)-y|\le\delta\}} \ge N\epsilon\bigg) \\ &\le \binom{N}{m} \sup_{\{u_j,y_j\}} \mathbb{P}\big(\big|Z_j^{(u_j)}(s) - y_j\big| \le \delta \ \ \forall j \le m\big). \end{aligned}\tag{6.19}$$

Here the supremum is over nonrandom $\{y_1, \dots, y_m\}$ (into which we incorporated the initial conditions $X_i(0)$) and all $[\inf\sigma, \sup\sigma]$-valued processes $\{u_1, u_2, \dots, u_m\}$ adapted to the filtration $\mathscr{H}_t$ generated by $\{W_i(r) : r \in [0,t], 1 \le i \le N\}$, while

$$Z_j^{(u_j)}(s) = \int_0^s u_j(r)\mathrm{d}W_j(r). \tag{6.20}$$

Each Itô integral in (6.20) is the L^2-limit of some $\mathscr{H}_t$-adapted stochastic integrals of processes that are piecewise constant in time. Therefore, we can and shall assume hereafter that $\{u_j\}$ are simple processes which are constant on each of the time intervals $[0, t_1), \dots, [t_{k-1}, t_k)$ for some partition $0 = t_0 < t_1 < \dots < t_k = s$ and $k \in \mathbb{N}$ (possibly dependent on N). The probability we maximize in (6.19) is then

$$\mathbb{E}\bigg[\prod_{j=1}^m G_\delta\bigg(\sum_{\ell=0}^{k-1} u_j(t_\ell)\Delta W_j(t_\ell), y_j\bigg)\bigg] \tag{6.21}$$

for $\Delta W_i(t_\ell) = W_i(t_{\ell+1}) - W_i(t_\ell)$ and $G_\delta(z, v) = \mathbb{1}_{|z-v|\le\delta}$. Such expectation, when conditioned upon

$$\Gamma_r = \{u_j(t_\ell), \Delta W_i(t_\ell), \ell \le k-2\} \cup \{u_j(t_{k-1}), \Delta W_i(t_{k-1}), i, j \ne r\}$$

becomes

$$\prod_{j\ne r} G_\delta(Z_j^{(u_j)}(s), y_j)\mathbb{E}\big[G_\delta(u_r(t_{k-1})\Delta W_r(t_{k-1}) + Z_r^{(u_r)}(t_{k-1}), y_r) \bigm| \Gamma_r\big].$$

The value of $\Delta W_r(t_{k-1})$ is independent of everything else, and clearly all that matters from Γ_r to the choice of $u_r(t_{k-1})$ that maximizes (6.21) is the value of $Z_r^{(u_r)}(t_{k-1})$. We thus conclude that the optimal $u_r(t_{k-1})$ may be assumed measurable with respect to the σ-algebra generated by $W_r(t)$, $t \in [0, t_{k-1}]$, and $u_r(t_\ell)$, $\ell \le k-2$. Substituting optimal $u_j(t_{k-1})$ of this type, for $1 \le j \le m$, changes the function G_δ considered in (6.21) but retains its form; namely, we need thereafter to maximize

$$\mathbb{E}\Big[\prod_{j=1}^m G'_\delta\big(Z_j^{(u_j)}(t_{k-1}), y_j\big)\Big]$$

for some (new) function $G'_\delta(z, v)$. The previous argument still applies, so proceeding by backward induction, from t_{k-1} to $t_{k-2}, \dots, t_0$, we conclude that it suffices to take the supremum in (6.19) only over $\{u_j : j \le m\}$ such that each process u_j is adapted to the filtration generated by W_j. The bound of (6.19) then becomes

$$\mathbb{P}(\rho^N \in B(\gamma, \delta)) \le \binom{N}{m}\Big[\sup_{u_1, y_1} \mathbb{P}\big(\big|Z_1^{(u_1)}(s) - y_1\big| \le \delta\big)\Big]^m, \tag{6.22}$$

where the supremum is now over nonrandom $y_1 \in \mathbb{R}$ and all $[\inf \sigma, \sup \sigma]$-valued processes u_1 adapted to the filtration generated by W_1. Since $m/N \ge \epsilon$ is bounded away from 0, we get (6.16) from

$$\limsup_{\delta\downarrow 0} \sup_{u_1, y_1} \mathbb{P}\big(\big|Z_1^{(u_1)}(s) - y_1\big| \le \delta\big) = 0,$$

which is an immediate consequence of the stronger result in [31, theorem 1].

Case (b). We continue with $b \equiv 0$, $\gamma(0, \cdot) = \rho_0$, and $R^{(\gamma)} \in C_b(\mathbb{R}_T)$, whereas (1.6) fails, namely

$$\int_0^T (\gamma(t), |x|^{1+\iota_\star})\mathrm{d}t = \infty \tag{6.23}$$

for $\iota_\star \in (0, 1]$ of Assumption 1.2(c). Let $0 \le f_K \uparrow f_\infty$ be infinitely differentiable functions such that

$$f_K(x) = |x|^{1+\iota_\star} \text{ on } [-K, -1] \cup [1, K], \quad f_K(x) \le 1 \text{ on } [-1, 1],$$
$$|f'_K|^2 \le 8f_K, \quad \|f_K\|_\infty \le 2K^{1+\iota_\star}, \quad \|f'_K\|_\infty \le 2K^{\iota_\star}, \quad \|f''_K\|_\infty \le 2$$

(we construct f_K by smoothing the function $(K \wedge (|x| \vee 1))^{1+\iota_\star}$ around the points $\{\pm 1, \pm K\}$). Next, for $Z_N^{f_K}(t) := (\rho^N(t), f_K)$ consider the stopping times

$$\tau_N^K(r) := \inf\{t \geq 0 : Z_N^{f_K}(t) \geq 2r\}.$$

Since $f_K(x) \leq 2|x|^{1+\iota_\star} + 1$ we have from Assumption 1.2(c) that for some $C_{\iota_\star}$ finite

$$\sup_{K,N \in \mathbb{N}} \{Z_N^{f_K}(0)\} \leq 2 \sup_{N \in \mathbb{N}} (\rho^N(0), |x|^{1+\iota_\star}) + 1 \leq C_{\iota_\star}. \tag{6.24}$$

With $\|f_K''\|_\infty \leq 2$ we get upon applying Itô's formula for $Z_N^{f_K}$ that, for the continuous martingale $M_N^{f_K}(t)$ of (6.3) and any $r \geq r_0 := C_{\iota_\star} + 2\|A\|_\infty T$,

$$\mathbb{P}\big(\tau_N^K(r) \leq T\big) \leq \mathbb{P}\big(M_N^{f_K}\big(\tau_N^K(r) \wedge T\big) \geq r\big). \tag{6.25}$$

Furthermore, since $|f_K'|^2 \leq 8 f_K$, we have for $V(\cdot)$ of (6.4) that

$$\langle M_N^{f_K}\rangle(t) = \frac{1}{N}\int_0^t V^{f_K}(\rho^N)(s)\mathrm{d}s \leq \frac{16\|A\|_\infty}{N}\int_0^t Z_N^{f_K}(s)\mathrm{d}s.$$

In particular, by the definition of $\tau_N^K(r)$,

$$\langle M_N^{f_K}\rangle\big(\tau_N^K(r) \wedge T\big) \leq \kappa \frac{r}{N} \tag{6.26}$$

for $\kappa = 128\|A\|_\infty T$. Appealing to the martingale representation theorem,

$$M_N^{f_K}\big(\tau_N^K(r) \wedge T\big) \overset{d}{=} \beta\big(\langle M_N^{f_K}\rangle\big(\tau_N^K(r) \wedge T\big)\big)$$

for some standard Brownian motion β. Hence, by (6.25) and (6.26), for any $r \geq r_0$,

$$\mathbb{P}\big(\tau_N^K(r) \leq T\big) \leq \mathbb{P}\Big(\sup_{t \in [0, \kappa r/N]} \{\beta(t)\} \geq r\Big) \leq 2\exp\left(-\frac{Nr}{2\kappa}\right). \tag{6.27}$$

Since d_L is a metric for the weak convergence in $M_1(\mathbb{R})$ and $f_K \in C_b(\mathbb{R})$, by dominated convergence the functionals $G_K(\xi) := \int_0^T (\xi(t), f_K)\mathrm{d}t$ on $\mathscr{C}$ are continuous with respect to the distance $d(\cdot,\cdot)$ of (1.5). Consequently, for any $K \in \mathbb{N}$, there exists $\delta_K > 0$ such that

$$\rho \in B(\gamma, \delta_K) \quad \Longrightarrow \quad G_K(\rho) \geq \frac{1}{2} G_K(\gamma).$$

Further, if $G_K(\rho^N) \geq 2Tr$, then necessarily $\tau_N^K(r) \leq T$. Thus, setting $r_K = \frac{1}{4T} G_K(\gamma)$, we have, for any K and $\delta \leq \delta_K$, that

$$\mathbb{P}(\rho^N \in B(\gamma, \delta)) \leq \mathbb{P}\big(G_K(\rho^N) \geq 2T r_K\big) \leq \mathbb{P}\big(\tau_N^K(r_K) \leq T\big),$$

which, in view of (6.27), yields the bound

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\rho^N \in B(\gamma, \delta)) \leq -\frac{G_K(\gamma)}{8\kappa T}$$

provided $G_K(\gamma) \geq 4Tr_0$. Our assumption (6.23) and the fact that $f_K(x) = |x|^{1+\iota_\star}$ for all $|x| \in [1, K]$ imply that $G_K(\gamma) \to \infty$ as $K \to \infty$, thereby establishing (6.16).

Case (c). Steps 1 and 2 of the proof of Proposition 6.1 require only that $\gamma(0) = \rho_0$ and $R^{(\gamma)} \in C_b(\mathbb{R}_T)$, both of which hold here. Hence, in view of the derivation leading to (6.13), we get (6.16) as soon as

$$\sup_{g \in \widetilde{\mathscr{S}}} I^g((\gamma, g)) = \infty \tag{6.28}$$

for $I^g(\cdot)$ of (6.15). Furthermore, $I^g((\gamma, g))$ is finite only if the identity

$$(\gamma, g)(t) = (\Psi h)(t) = (\gamma, g)(0) + \overline{H}^g(\gamma)(t) + h(\overline{V}^g(\gamma)(t)) \tag{6.29}$$

holds for some h absolutely continuous. By assumption, there exists $g \in \widetilde{\mathscr{S}}$ for which the left-hand side of (6.29) is not absolutely continuous on $[0, T]$. In contrast, both $\overline{H}^g(\gamma)(\cdot)$ and $\overline{V}^g(\gamma)(\cdot)$ are absolutely continuous for any g and γ, hence so is the right-hand side of (6.29) for any absolutely continuous h, resulting with (6.28). □

We conclude by showing the exponential tightness of $\{\rho^N : N \in \mathbb{N}\}$.

PROPOSITION 6.3. *Under Assumption* 1.2(a)–(c), *the sequence* $\{\rho^N\}$ *is exponentially tight on* $\mathscr{C}$. *That is, for any finite* M *there exists a compact* $K_M \subset \mathscr{C}$ *for which*

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}\big(\rho^N \notin K_M\big) \leq -M. \tag{6.30}$$

PROOF. In view of (6.12) it suffices to confirm the criterion for exponential tightness given in [9, lemma A.2]. Specifically, this amounts to showing as Step 1 that $\{\rho^N(t) : N \in \mathbb{N}\}$ is exponentially tight on $(M_1(\mathbb{R}), d_L)$ for each fixed $t \in [0, T]$ (rational), and then proving as Step 2 that, for any fixed $\epsilon > 0$, one has

$$\lim_{\kappa \downarrow 0} \sup_{N \in \mathbb{N}} \frac{1}{N} \log \mathbb{P}\Big(\sup_{\substack{0 \leq s,t \leq T, \\ |t-s| \leq \kappa}} d_L(\rho^N(s), \rho^N(t)) > \delta\Big) = -\infty. \tag{6.31}$$

Step 1. By Prokhorov's theorem, the set

$$\{\alpha \in M_1(\mathbb{R}) : (\alpha, \phi) \leq C\}$$

is precompact in $(M_1(\mathbb{R}), d_L)$ for $\phi(x) = |x|$ and any C finite. Hence, to prove our first assertion, it suffices to show that for some $C = C(M, T) < \infty$

$$\limsup_{N \to \infty} \frac{1}{N} \log \sup_{t \in [0,T]} \mathbb{P}((\rho^N(t), \phi) > 2C) \leq -M. \tag{6.32}$$

To this end, from (6.24) we have that $\sup_N (\rho^N(0), \phi) \leq C_0$ finite, hence with

$$Z_i(t) = \int_0^t \sigma\big(F_{\rho^N(r)}(X_i(r))\big) \mathrm{d}W_i(r), \tag{6.33}$$

it follows by Markov's inequality, that for any $C \geq C_0 + T\|b\|_\infty$

$$\mathbb{P}((\rho^N(t), \phi) > 2C) = \mathbb{P}\left(\sum_{i=1}^N |X_i(t)| > 2CN\right)$$

$$\leq \mathbb{P}\left(\sum_{i=1}^N |Z_i(t)| > CN\right) \leq e^{-CN} \sup_{\{u_j\}} \mathbb{E}\left[\prod_{j=1}^N e^{|Z_j^{(u_j)}(t)|}\right]. \tag{6.34}$$

The supremum here is over the same collection of simple adapted processes $\{u_j\}$ we considered in (6.21) and with $Z_j^{(u_j)}(t)$ as in (6.20). By the same argument we have used en route to (6.22), it suffices to consider the situation where each process u_j is adapted to the filtration generated by the Brownian motion W_j. Consequently, the preceding upper bound simplifies to

$$\mathbb{P}((\rho^N(t), \phi) > 2C) \leq e^{-CN} \sup_{u_1} \mathbb{E}\big[e^{|Z_1^{(u_1)}(t)|}\big]^N, \tag{6.35}$$

where the supremum is now over all $[\inf \sigma, \sup \sigma]$-valued processes u_1 adapted to the filtration generated by W_1. Viewing such a process $Z_1^{(u_1)}(t)$ as a time-changed standard Brownian motion β results in

$$\sup_{u_1} \mathbb{E}\big[e^{|Z_1^{(u_1)}(t)|}\big] \leq \mathbb{E}\big[\exp(\sup_{r\in[0,2t\|A\|_\infty]} |\beta(r)|)\big] < \infty,$$

which together with (6.35) yields (6.32).

Step 2. First note that for $d_{BL}(\cdot,\cdot)$ of (6.12), $Z_i(t)$ of (6.33), and any $s, t \in [0, T]$,

$$\begin{aligned}
\frac{1}{2} d_L(\rho^N(s), \rho^N(t))^2 &\leq d_{BL}(\rho^N(s), \rho^N(t)) \\
&\leq \frac{1}{N} \sum_{i=1}^N |X_i(t) - X_i(s)| \\
&\leq \frac{1}{N} \sum_{i=1}^N |Z_i(t) - Z_i(s)| + |t-s| \|b\|_\infty
\end{aligned} \tag{6.36}$$

(see [13, proof of theorem 11.3.3] for the leftmost inequality). Thus, we have (6.31) as soon as we show that, for

$$\mathsf{osc}(Z; \kappa, T) = \sup_{\substack{0\leq s,t\leq T,\\ |t-s|\leq\kappa}} \{|Z(t) - Z(s)|\}$$

and any fixed $\lambda, \delta > 0$,

$$\lim_{\kappa\downarrow 0} \sup_{N\in\mathbb{N}} \frac{1}{N} \log \mathbb{P}\left(\sum_{i=1}^N \mathsf{osc}(Z_i; \kappa, T) > \delta N\right) \leq -\lambda\delta. \tag{6.37}$$

As for the proof of (6.37), similarly to the derivation of (6.34) and (6.35), by Markov's inequality we have that

$$\mathbb{P}\Big(\sum_{i=1}^{N} \mathsf{osc}(Z_i;\kappa,T) > \delta N\Big) \le e^{-\lambda\delta N} \sup_{\{u_j\}} \mathbb{E}\Big[\prod_{j=1}^{N} e^{\lambda\,\mathsf{osc}(Z_j^{(u_j)};\kappa,T)}\Big]$$

$$\le e^{-\lambda\delta N}\big(\sup_{u_1}\mathbb{E}\big[e^{\lambda\,\mathsf{osc}(Z_1^{(u_1)};\kappa,T)}\big]\big)^{N}, \tag{6.38}$$

where the latter supremum is over all $[\inf\sigma,\sup\sigma]$-valued processes u_1 adapted to the filtration generated by W_1. In addition, viewing the continous martingale $Z_1^{(u_1)}(t)$ as a time-changed standard Brownian motion β, it follows by the same reasoning we applied in Step 1 that

$$\sup_{u_1} \mathbb{E}\big[e^{\lambda\,\mathsf{osc}(Z_1^{(u_1)};\kappa,T)}\big] \le \mathbb{E}\big[e^{\lambda\,\mathsf{osc}(\beta;2\|A\|_\infty\kappa,2\|A\|_\infty T)}\big]. \tag{6.39}$$

Now, since

$$\mathsf{osc}(\beta;2\|A\|_\infty\kappa,2\|A\|_\infty T) \le 2\sup_{r\in[0,2\|A\|_\infty T]}\{|\beta(r)|\},$$

which has finite exponential moments of all orders, by sample path continuity of β and dominated convergence, the right-hand side of (6.39) decays to 1 as $\kappa\downarrow 0$. Thus, combining (6.39) and (6.38) results in (6.37), thereby completing the proof of the proposition. □

7 Proof of Proposition 2.7

We work throughout under parts (a)–(c) of Assumption 1.2 with $\iota_\star = 1$.

7.1 Proof of Part (a)

From [2, sec. 3] we have that in any solution of (1.1) the ordered particles $X_{(1)}(t) \le X_{(2)}(t) \le \cdots \le X_{(N)}(t)$ satisfy the SDS

$$\mathrm{d}X_{(j)}(t) = b_j\,\mathrm{d}t + \sigma_j\,\mathrm{d}\beta_j(t) + \frac{1}{2}\,\mathrm{d}\Lambda_{j-1}(t) - \frac{1}{2}\,\mathrm{d}\Lambda_j(t), \quad j=1,\dots,N, \tag{7.1}$$

for independent standard Brownian motions $\{\beta_j\}$ where $\Lambda_0(t)=\Lambda_N(t)\equiv 0$ and $\Lambda_j(t)$ for $j=1,\dots,N-1$ denotes the local time at 0 accumulated by the $\mathbb{R}_+$-valued path $X_{(j+1)}(\cdot)-X_{(j)}(\cdot)$ by time t. It is also shown in [2, sec. 3] that strong existence and uniqueness holds for the SDS (7.1). We claim that, with probability 1,

$$t \mapsto \Delta_N(t) := \frac{1}{N}\sum_{j=1}^{N}\big|X_{(j)}(t)-\widetilde{X}_{(j)}(t)\big| \tag{7.2}$$

is nonincreasing for any two strong solutions X and $\widetilde{X}$ of (7.1) driven by the same Brownian motions $\{\beta_j\}$ (extending [22, inequality (15)] to arbitrary initial conditions). Indeed, since $t\mapsto (X_{(j)}(t)-\widetilde{X}_{(j)}(t))$ is of bounded variation, its local time process at 0 vanishes. Thus, setting $S_j(t) := \operatorname{sgn}(X_{(j)}(t) - \widetilde{X}_{(j)}(t))$, we have by Tanaka's formula, followed by summation by parts, that

$$\begin{aligned} \mathrm{d}\Delta_N(t) &= \frac{1}{2N}\sum_{j=1}^{N} S_j(t)\big(\mathrm{d}\Lambda_{j-1}(t) - \mathrm{d}\Lambda_j(t) - \mathrm{d}\widetilde{\Lambda}_{j-1}(t) + \mathrm{d}\widetilde{\Lambda}_j(t)\big) \\ &= \frac{1}{2N}\sum_{j=2}^{N} (S_j(t) - S_{j-1}(t))\big(\mathrm{d}\Lambda_{j-1}(t) - \mathrm{d}\widetilde{\Lambda}_{j-1}(t)\big). \end{aligned} \tag{7.3}$$

Because $X_{(j)}(t) = X_{(j-1)}(t)$ at times of increase of $\Lambda_{j-1}(t)$ and $\widetilde{X}_{(j)}(t) = \widetilde{X}_{(j-1)}(t)$ at times of increase of $\widetilde{\Lambda}_{j-1}(t)$, it is easy to check, for $j = 2, \dots, N$ and all $t \geq 0$, that

$$(S_j(t) - S_{j-1}(t))\mathrm{d}\Lambda_{j-1}(t) \leq 0 \leq (S_j(t) - S_{j-1}(t))\mathrm{d}\widetilde{\Lambda}_{j-1}(t).$$

Hence, by (7.3) we have, as claimed, that $\mathrm{d}\Delta_N(t) \leq 0$. Next, with ρ^N and $\widetilde{\rho}^N$ denoting the paths of empirical measures of X and $\widetilde{X}$, respectively, we see that for any $t \geq 0$,

$$d_{BL}(\rho^N(t), \widetilde{\rho}^N(t)) \leq \Delta_N(t) \leq \Delta_N(0) = W_1(\rho^N(0), \widetilde{\rho}^N(0)), \tag{7.4}$$

where $W_1(\cdot,\cdot)$ stands for the L_1-Wasserstein distance on $M_1(\mathbb{R})$:

$$W_1(\alpha_1, \alpha_2) := \inf\{\mathbb{E}[|Y_1 - Y_2|] : Y_1 \sim \alpha_1, Y_2 \sim \alpha_2\}.$$

Further, as $\frac{1}{2}d_L(\alpha_1, \alpha_2)^2 \leq d_{BL}(\alpha_1, \alpha_2)$ (see (6.36)), it follows from (7.4) that

$$\frac{1}{2}d(\rho^N, \widetilde{\rho}^N)^2 \leq W_1(\rho^N(0), \widetilde{\rho}^N(0)).$$

Fixing ℓ, ϵ we let $\rho^{N,\ell,\epsilon}(0)$ be the empirical measure of $\widetilde{X}_{(j)}(0) = F^{-1}(u_{j,N})$, with $u_{j,N} = j/(N+1)$ and $F := F_{\gamma^{\ell,\epsilon}(0)}$ a continuous CDF. By the preceding, we get (2.15) upon showing that

$$\limsup_{\epsilon\to 0}\limsup_{\ell\to\infty}\limsup_{N\to\infty} W_1(\rho^N(0), \rho^{N,\ell,\epsilon}(0)) = 0. \tag{7.5}$$

We then complete the proof of part (a) by observing that

$$\begin{aligned} \frac{N}{N+1}(|x|, \rho^{N,\ell,\epsilon}(0)) &= \frac{1}{N+1}\sum_{j=1}^{N}\big|F^{-1}(u_{j,N})\big| \\ &\leq \int_0^1 |F^{-1}(u)|\mathrm{d}u = (|x|, \gamma^{\ell,\epsilon}(0)), \end{aligned} \tag{7.6}$$

which is finite since $\gamma^{\ell,\epsilon}(0) \in M_1^{(0)}(\mathbb{R})$, whereas for $N \to \infty$ and any fixed $x \in \mathbb{R}$,

$$\rho^{N,\ell,\epsilon}(0)((-\infty, x]) = \frac{1}{N}\lfloor (N+1)F(x)\rfloor \to F(x).$$

That is, as claimed our $\{\rho^{N,\ell,\epsilon}(0)\}$ satisfy Assumption 1.2(c) (with $\iota_\star = 0$). In particular,

$$\lim_{N\to\infty} d_{BL}(\rho^{N,\ell,\epsilon}(0), \gamma^{\ell,\epsilon}(0)) = 0. \tag{7.7}$$

Turning to proving (7.5), recall that, for any $\kappa \geq 1$,

$$W_1(\alpha_1, \alpha_2) \leq \kappa d_{BL}(\alpha_1, \alpha_2) + 3(|x|\mathbb{1}_{|x|>\kappa}, \alpha_1 + \alpha_2) \tag{7.8}$$

(for example, this follows from [13, theorem 11.8.2]). From Assumption 1.2(c) we know that $d_{BL}(\rho^N(0), \gamma(0)) \to 0$ when $N \to \infty$, whereas $d_{BL}(\gamma(0), \gamma^{\ell,\epsilon}(0)) \to 0$ when $\ell \to \infty$ followed by $\epsilon \to 0$ (see (2.11)). Combining these facts with (7.7), (7.8), and the triangle inequality for $d_{BL}(\cdot,\cdot)$, we get (7.5) by showing that

$$\lim_{\kappa\to\infty} \limsup_{N\to\infty} \big(|x|\mathbb{1}_{|x|>\kappa}, \rho^N(0)\big) = 0, \tag{7.9}$$

$$\lim_{\kappa\to\infty} \limsup_{\epsilon\to 0} \limsup_{\ell\to\infty} \limsup_{N\to\infty} (|x|\mathbb{1}_{|x|>\kappa}, \rho^{N,\ell,\epsilon}(0)) = 0. \tag{7.10}$$

With $\sup_N(|x|^2, \rho^N(0))$ finite (see Assumption 1.2(c), $\iota_\star = 1$), we obviously have (7.9). As for (7.10), from (7.7) and $\gamma^{\ell,\epsilon}(0)$ having a density, we deduce that as $N \to \infty$

$$\big(|x|\mathbb{1}_{|x|\leq\kappa}, \rho^{N,\ell,\epsilon}(0)\big) \to \big(|x|\mathbb{1}_{|x|\leq\kappa}, \gamma^{\ell,\epsilon}(0)\big).$$

Further, when $\ell \to \infty$, $\epsilon \to 0$, and then $\kappa \to \infty$, we have by (2.10) that

$$\big(|x|\mathbb{1}_{|x|>\kappa}, \gamma^{\ell,\epsilon}(0)\big) \leq \frac{1}{\ell}(|x|, \widehat{\mu}^\epsilon) + \frac{1}{\kappa}(|x|^2, \gamma^\epsilon(0)) \to 0,$$

which together with (7.6) yields (7.10).

7.2 Proof of Part (b)

From part (a) we know that Assumption 1.2(c) holds for $\iota_\star = 0$, $\rho^{N,\ell,\epsilon}(0)$, and $\rho_0^{\ell,\epsilon} = \gamma^{\ell,\epsilon}(0)$, so we simplify notation by dropping hereafter the superscripts ℓ, ϵ. That is, we fix $\gamma \in \mathscr{G}$ with $\widetilde{J}(\gamma) < \infty$ (see Definition 2.4 and (1.8), respectively), having $R = R^{(\gamma)} \in C_b^\infty(\mathbb{R}_T) \bigcap \mathscr{F}_{3/2}$ (for $\mathscr{F}_q$ of (1.7)), starting at $R(0, x) = F_{\rho_0}(x)$, with R_x strictly positive on $\mathbb{R}_T$ and such that

$$h = \frac{R_t - (A(R)R_x)_x}{A(R)R_x} \in C_b(\mathbb{R}_T) \tag{7.11}$$

with $x \mapsto h(t, x)$ uniformly Lipschitz continuous on $\mathbb{R}_T$. The functional on $\mathscr{C}$

$$\widehat{J}(\rho) = \frac{1}{2}\int_0^T (\rho, U(\rho)^2)(s)\mathrm{d}s, \tag{7.12}$$

with the continuous-in-time, bounded function on $\mathbb{R}_T$

$$U(\rho) := \frac{hA(R^{(\rho)}) + b(R^{(\rho)})}{\sigma(R^{(\rho)})},$$

is then such that $\widetilde{J}(\gamma) = \widehat{J}(\gamma) < \infty$.

To prove the local large deviations lower bound (2.16) we introduce in Step 1 a suitable change of measure to $\mathbb{Q}_1$ for which the relevant Radon-Nikodym derivative is at least $e^{-N(\widehat{J}(\gamma)+\epsilon)}$ in the event $\{\rho^N \in B(\gamma,\delta)\}$ (when $N\to\infty$ followed by $\delta \downarrow 0$), then verify in Step 2 the relevant LLN (7.16) for ρ^N under $\mathbb{Q}_1$. This step relies on proving in Lemma 7.1 that any limit point $\widehat{\gamma}$ of ρ^N has no atoms, from which we deduce that the corresponding path of CDFs $w = R^{(\widehat{\gamma})}$ solves the weak form (7.18) of the porous medium equation (1.10). The tilt h in the latter equation is given by (7.11), so $R^{(\gamma)}$ is one such solution, and the analysis of Lemma 7.2 guarantees its uniqueness.

Step 1. Since $x \mapsto h(t,x)$ is a uniformly bounded, Lipschitz function on $\mathbb{R}_T$, there exists, for any $q\in\mathbb{R}$, a probability measure $\mathbb{Q}_q$ under which, for $i = 1,\dots,N$,

$$X_i(t) = X_i(0) + \int_0^t \psi_q(\rho^N)(s, X_i(s))\mathrm{d}s + \int_0^t \sigma\big(F_{\rho^N(s)}(X_i(s))\big)\mathrm{d}W_i^q(s), \tag{7.13}$$

with $\psi_q(\rho) = b(R^{(\rho)}) - qU(\rho)\sigma(R^{(\rho)})$ and $\{W_i^q\}$ independent standard Brownian motions. Furthermore, $\mathbb{P} = \mathbb{Q}_0$ and, similarly to (6.17) (which has $h\equiv 0$, i.e. $\psi_q = (1-q)b(R^{(\rho)})$), by Girsanov's theorem

$$\frac{\mathrm{d}\mathbb{Q}_q}{\mathrm{d}\mathbb{P}} = \exp\left(-qM_N(T) - \frac{q^2}{2}\langle M_N\rangle(T)\right),$$

with the continuous martingales

$$M_N(t) = \sum_{i=1}^N \int_0^t U(\rho^N)(s, X_i(s))\,\mathrm{d}W_i^0(s)$$

satisfying $\langle M_N\rangle(T) = 2N\widehat{J}(\rho^N)$ (see [24, theorem 3.5.1]). By the triangle inequality, for any $\rho\in B(\gamma,\delta)$,

$$2|\widehat{J}(\rho) - \widehat{J}(\gamma)| \le \sup_{\rho\in B(\gamma,\delta)}\left|\int_0^T (\rho, U(\rho)^2 - U(\gamma)^2)(s)\mathrm{d}s\right| + \sup_{\rho\in B(\gamma,\delta)}\left|\int_0^T (\rho, U(\gamma)^2)(s)\mathrm{d}s - \int_0^T (\gamma, U(\gamma)^2)(s)\mathrm{d}s\right|. \tag{7.14}$$

Since b and σ are bounded, Lipschitz functions, and h,σ^{-1} are uniformly bounded, it is easy to check that $U(\rho)^2$ is a bounded, Lipschitz function of $R^{(\rho)}$. Consequently, $|U(\rho)^2 - U(\gamma)^2| \le C|R^{(\rho)} - R^{(\gamma)}|$ for some finite C and all $(t,x)\in\mathbb{R}_T$. Thus, from (6.8) it follows that the first term on the right-hand side of (7.14) converges to 0 as $\delta\downarrow 0$. Further, with h and $R^{(\gamma)}$ in $C_b(\mathbb{R}_T)$, also $U(\gamma)^2 \in C_b(\mathbb{R}_T)$.

Hence, by dominated convergence, the functional $\xi \mapsto \int_0^T (\xi, U(\gamma)^2)(s)\mathrm{d}s$ is continuous on $\mathscr{C}$, and the second term on the right-hand side of (7.14) also converges to 0 as $\delta \downarrow 0$. We conclude that, for any fixed $\epsilon > 0$ and all $\delta < \delta_0(\epsilon)$,

$$\rho^N \in B(\gamma, \delta) \quad \Longrightarrow \quad \left| \frac{1}{2}\langle M_N \rangle(T) - N\widehat{J}(\gamma) \right| \le \epsilon N. \tag{7.15}$$

For such $\delta < \delta_0(\epsilon)$, any $p = q/(q-1) > 1$, $q > 1$, and all $N \in \mathbb{N}$, we thus have by Hölder's inequality that

$$\begin{aligned}
&\mathbb{Q}_1(\rho^N \in B(\gamma, \delta)) \\
&\quad = \mathbb{P}\big[e^{-M_N(T) - \frac{1}{2}\langle M_N\rangle(T)} \mathbb{1}_{\{\rho^N \in B(\gamma,\delta)\}}\big] \\
&\quad \le \mathbb{P}\big[e^{-qM_N(T) - \frac{q}{2}\langle M_N\rangle(T)} \mathbb{1}_{\{\rho^N \in B(\gamma,\delta)\}}\big]^{1/q} \mathbb{P}(\rho^N \in B(\gamma,\delta))^{1/p} \\
&\quad \le e^{(q-1)N(\widehat{J}(\gamma)+\epsilon)} \mathbb{P}\big[e^{-qM_N(T) - \frac{q^2}{2}\langle M_N\rangle(T)}\big]^{1/q} \mathbb{P}(\rho^N \in B(\gamma,\delta))^{1/p} \\
&\quad = e^{(q-1)N(\widehat{J}(\gamma)+\epsilon)} [\mathbb{Q}_q(1)]^{1/q} \mathbb{P}(\rho^N \in B(\gamma,\delta))^{1/p}.
\end{aligned}$$

Consider $\frac{p}{N}\log(\cdot)$ of both sides, taking first $N \to \infty$ followed by $\delta \downarrow 0$, to find that

$$\lim_{\delta \downarrow 0} \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\rho^N \in B(\gamma,\delta)) \ge -q(\widehat{J}(\gamma) + \epsilon)$$

for any $\epsilon > 0$, $q > 1$ (hence (2.16) holds), provided that

$$\lim_{N \to \infty} \mathbb{Q}_1(\rho^N \in B(\gamma,\delta)) = 1. \tag{7.16}$$

Step 2. To prove (7.16), recall that $\mathbb{Q}_1$ corresponds to $\{X_i(t)\}$ of (7.13) for drift $\psi_1(\rho) = -hA(R^{(\rho)})$ and, for each $N \in \mathbb{N}$, let Q_N denote the law of ρ^N under $\mathbb{Q}_1$. One then deduces the uniform tightness, and hence precompactness, of the collection $\{Q_N\}$ in the space of probability measures on $\mathscr{C}$. Indeed, Steps 1 and 2 of the proof of [36, theorem 1.1] rely only on a general compactness criterion for subsets of $\mathscr{C}$ (taken from [16, lemma 1.3]), so they can be carried out mutatis mutandis, appealing here to boundedness of the drift and diffusion coefficients in (7.13) (in place of boundedness of the corresponding coefficients in the dynamics treated in [36, theorem 1.1]). Moreover, since $\rho^N(0) \to \rho_0 = \gamma(0,\cdot)$, the computations there that involve the initial conditions can be omitted here. To prove (7.16), it thus suffices to show that the atomic measure δ_γ is the only possible limit point of the sequence $\{Q_N\}$. Alternatively, passing to the relevant subsequence and utilizing the Skorokhod representation theorem in the form of [12, theorem 3.5.1], we can and shall assume that the variables ρ^N, $N \in \mathbb{N}$, are defined on the same probability space and converge almost surely in $\mathscr{C}$, when $N \to \infty$, to some limiting variable $\widehat{\gamma}$. Thus, the task of proving (7.16) amounts to showing that $\widehat{\gamma} = \gamma$ with probability 1.

To this end, fixing $g \in \widetilde{\mathscr{S}}$ and replacing hereafter $b(R^{(\xi)})$ by $-hA(R^{(\xi)})$ in the definition of $\mathscr{R}^\xi g$, note that the left-hand side of the identity (6.2) converges with probability 1, as $N \to \infty$ to $(\widehat{\gamma}, g)(t) - (\rho_0, g(0))$ for all $t \in [0, T]$. We claim that with probability 1 the right-hand side of (6.2) converges to $\int_0^t H^g(\widehat{\gamma})(s)\mathrm{d}s$ and consequently for all $t \in [0, T]$,

$$(7.17) \qquad (\widehat{\gamma}, g)(t) - (\rho_0, g(0)) = \int_0^t (\widehat{\gamma}, g_t + (g_{xx} - hg_x)A(R^{(\widehat{\gamma})}))(s)\mathrm{d}s.$$

Indeed, recall that $\overline{V}^g(\rho^N)(T)$ is uniformly bounded by $C = 2T\|Ag_x^2\|_\infty$ finite, so $\langle M_N^g\rangle(T) \leq C/N$ and, with probability 1, the continuous martingale $M_N^g(t) \to 0$ uniformly over $[0, T]$ (for example, combine the Burkholder-Davis-Gundy inequality, as in [24, theorem 3.3.28], with the Borel-Cantelli lemma). Furthermore, we show in Lemma 7.1 that with probability 1 $R^{(\widehat{\gamma})} \in C_b(\mathbb{R}_T)$, from which one deduces as in the derivation of (6.8), that with probability 1

$$\lim_{N\to\infty} \int_0^T (\rho^N, |F_{\rho^N} - R^{(\widehat{\gamma})}|)(s)\mathrm{d}s = 0.$$

Since $R^{(\widehat{\gamma})} \in C_b(\mathbb{R}_T)$, also $\mathscr{R}^{\widehat{\gamma}} g \in C_b(\mathbb{R}_T)$. Thus, with hg_x and g_{xx} bounded, the preceding convergence to 0 implies, as in the derivation of (6.10), that with probability 1

$$\lim_{N\to\infty} \int_0^T |H^g(\rho^N)(s) - H^g(\widehat{\gamma})(s)|\mathrm{d}s = 0,$$

thereby completing the proof of (7.17).

Now, setting $w = R^{(\widehat{\gamma})}$ and having (7.17) hold with probability 1 for all g in a countable dense subset of $\widetilde{\mathscr{S}}$, we deduce that (7.17) holds for all $g \in \widetilde{\mathscr{S}}$, which amounts after integration by parts over $\mathbb{R}$ to

$$(7.18) \qquad \int_{\mathbb{R}} (fw)(t, x)\mathrm{d}x - \int_{\mathbb{R}} (fw)(0, x)\mathrm{d}x = \int_{\mathbb{R}_t} [f_t w + f_{xx}\Sigma(w) - (hf)_x \Sigma(w)]\mathrm{d}m,$$

holding for $w(0, x) = F_{\rho_0}(x)$ and all $f = g_x \in \widetilde{\mathscr{S}}_x$. Here $\Sigma(w) = \int_0^w A(r)\mathrm{d}r$ and h_x is well-defined (m-a.e.), since $x \mapsto h(t, x)$ is a Lipschitz function. Note that, for $w \in C^{1,2}(\mathbb{R}_T)$, further integration by parts (to eliminate all derivatives of f) confirms that this is equivalent to w solving the porous medium equation (1.10) (with initial condition F_{ρ_0}). In view of (7.11) one such solution is $w = R^{(\gamma)}$; hence with probability 1, $\widehat{\gamma} = \gamma$ provided we establish the uniqueness of such a generalized solution. In conclusion, all that remains for establishing Proposition 2.7 is to prove the following two lemmas.

LEMMA 7.1. *Consider for $\psi_1(\rho) = -hA(R^{(\rho)})$ the unique weak solution of* (7.13)*, and suppose that $\widehat{\gamma}$ is an a.s. limit point in $\mathscr{C}$ of ρ^N. Then, with probability* 1 *the probability measures $\widehat{\gamma}(t, \cdot)$ have no atoms for all $t \in [0, T]$.*

LEMMA 7.2. *Suppose $h \in C_b(\mathbb{R}_T)$ with $x \mapsto h(t,x)$ uniformly Lipschitz on $\mathbb{R}_T$, for which* (1.10) *has a bounded classical solution $R = u \in C^{1,2}(\mathbb{R}_T)$ with $u(0,x) = F_{\rho_0}(x)$. It is then the only solution $w \in C_b(\mathbb{R}_T)$ of* (7.18) *with such initial conditions for which $w(t,\cdot)$ are CDFs of some path in $\mathscr{C}$.*

PROOF OF LEMMA 7.1. When proving Proposition 6.2, we first removed the drift $b(R^{(\rho)})$ thanks to the bound (6.18), which applies for any uniformly bounded drift. Using the same argument for the measure $\mathbb{Q}_1$ with bounded drift $-hA(R^{(\rho)})$, we conclude as in case (a) of the proof of Proposition 6.2 that, for any $\epsilon > 0$ and some $\delta = \delta(\epsilon) \in (0,\epsilon)$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{\substack{s\in[0,T],\\ y\in\mathbb{R}}} \mathbb{Q}_1(\rho^N(s)([y-3\delta, y+3\delta]) \geq \epsilon) \leq -1. \tag{7.19}$$

Similarly, Step 1 and Step 2 of the proof of Proposition 6.3 apply whenever the drift is uniformly bounded; hence, for any $\delta, \epsilon > 0$, some $M = M(\epsilon)$ finite, and $\kappa(\delta) > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{s\in[0,T]} \mathbb{Q}_1(\rho^N(s)([-M,M]) \leq 1-\epsilon) \leq -1, \tag{7.20}$$

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{Q}_1\Big(\sup_{\substack{0\leq s,t\leq T,\\ |t-s|\leq\kappa}} d_L(\rho^N(s), \rho^N(t)) > \delta\Big) \leq -1 \tag{7.21}$$

(see (6.32) and (6.31), respectively). Fixing $\delta(\epsilon) > 0$ and $M(\epsilon)$ finite, then $\kappa(\delta) > 0$, consider (7.19) and (7.20) at all $\{s_i : i \leq m\}$ on a finite κ-net in $[0,T]$ and all $\{y_j : j \leq \ell\}$ in a finite δ-net within $[-(M+2\delta), (M+2\delta)]$ to conclude by (7.21) and the Borel-Cantelli lemma that with probability 1 for all N large enough,

$$\sup_{1\leq i\leq m} \sup_{y\in\mathbb{R}} \{\rho^N(s_i)([y-2\delta, y+2\delta])\} \leq \epsilon,$$
$$\sup_{t\in[0,T]} \inf_{1\leq i\leq m} \big\{d_L\big(\rho^N(s_i), \rho^N(t)\big)\big\} \leq \delta.$$

Consequently, as in (6.9),

$$\sup_{t\in[0,T]} \sup_{y\in\mathbb{R}} \{\rho^N(t)([y-\delta, y+\delta])\} \leq 3\epsilon.$$

Combining this with the Portmanteau theorem, we infer that, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any limit point $\widehat{\gamma}$ of ρ^N, with probability 1,

$$\begin{aligned}
&\sup_{t\in[0,T]} \sup_{y\in\mathbb{R}} \widehat{\gamma}(t)((y-\delta, y+\delta)) \\
&\quad \leq \sup_{t\in[0,T]} \sup_{y\in\mathbb{R}} \liminf_{N\to\infty} \rho^N(t)((y-\delta, y+\delta)) \\
&\quad \leq \limsup_{N\to\infty} \sup_{t\in[0,T]} \sup_{y\in\mathbb{R}} \{\rho^N(t)((y-\delta, y+\delta))\} \leq 3\epsilon.
\end{aligned}$$

This shows that with probability 1 the path of measures $\widehat{\gamma}$ has no atoms of mass at least 3ϵ, and taking $\epsilon \downarrow 0$ finishes the proof of the lemma. $\square$

PROOF OF LEMMA 7.2. First, note that if (7.18) holds for $w(t,\cdot)$ which are the CDFs of some path in $\mathscr{C}$ and all $f \in \widetilde{\mathscr{S}_x}$, then it further holds for all $\widehat{f} \in \widetilde{\mathscr{S}}$. Indeed, for any such $\widehat{f}$ there exist $f_k \in \widetilde{\mathscr{S}_x}$ coinciding with $\widehat{f}$ on $[0,T]\times[-k,\infty)$ such that both $\sup_k \|f_k - \widehat{f}\|_{W_1^{1,2}(\mathbb{R}_T)}$ and $\sup_{k,t} \|f_k(t,\cdot) - \widehat{f}(t,\cdot)\|_{L^1(\mathbb{R})}$ are finite. Thus, with $x \mapsto h(x,t)$ uniformly bounded, globally Lipschitz on $\mathbb{R}_T$ and $\Sigma(w) \le \|A\|_\infty w$, it follows from (3.1) that the value each side of (7.18) takes for f_k converges as $k \to \infty$ to its value for $\widehat{f}$, thereby extending the scope of (7.18) to all of $\widetilde{\mathscr{S}}$. The latter identity involves only (f, f_t, f_x, f_{xx}) and hence holds for any $f \in C_c^{1,2}(\mathbb{R}_T)$. In addition, setting $\mathbb{K}_r = [0,t]\times[-r,r]$, the identity (7.18) applies for any $f \in C^{1,2}(\mathbb{K}_r)$ such that $f(s,\pm r) = 0$ for all $s \in [0,t]$ provided one adds to its left-hand side the boundary term

$$\int_0^t [(\Sigma(w) f_x)(s,r^-) - (\Sigma(w) f_x)(s,-r^-)]\mathrm{d}s$$

(which takes into account the jump discontinuity of f_x at $\partial\mathbb{K}_r$). That is, $w(t,x)$ is a generalized solution of the (CP) problem as in [11, def. 1.1] except for replacing the term $f_x b(w)$ there by $(hf)_x\Sigma(w)$. The uniqueness of such nonnegative $w \in C_b(\mathbb{R}_T)$, starting at the nonnegative $w(0,x) = u(0,x) \in C_b(\mathbb{R})$, thus follows by adapting the proof of [11, theorem 4.2 (1)] to handle $h \not\equiv 1$.

To this end, we modify hereafter (7.18) as above and prove the analogue of [11, (4.2)]. That is, we call w a subsolution if the right-hand side of (7.18) is greater than or equal to its (modified) left-hand side for every nonnegative $f \in C^{1,2}(\mathbb{K}_r)$, any $r > 0$, and all $t \in (0,T]$, while w is a supersolution when the corresponding left-hand side is greater than or equal to the right-hand side for any such f, r, t. It then suffices to show that for some $c = c(h)$ finite, any supersolution w, all $t \in [0,T]$, $\ell \in (0,\infty)$, and $[0,1]$-valued $\omega \in C_c^\infty([-\ell,\ell])$,

$$\int_{\mathbb{R}} (u(t,x) - w(t,x))\omega(x)\mathrm{d}x \le c \int_{\mathbb{R}} (u(0,x) - w(0,x))_+ \,\mathrm{d}x, \tag{7.22}$$

where the same inequality holds with $u - w$ replaced by $w - u$ for any subsolution w. Indeed, by definition of a supersolution (or subsolution), the right-hand side of (7.22) is 0. Hence, choosing $\omega(x)$ as smooth approximations of $\mathbb{1}_{u(t,x)>w(t,x)}$ on $(-\ell,\ell)$ and sending $\ell \to \infty$, we deduce that m-a.e. $u \le w$ for supersolutions and $u \ge w$ for subsolutions, yielding the stated uniqueness of the solution w. Fixing t, ℓ, and ω, we prove (7.22) for a given supersolution w (exchanging the roles of u and w then yields the proof for subsolutions). As in the proof of [11, (4.2)], for $f = f^{(n,r)}$, $n, r \ge \ell + 1$ that solve suitable linear parabolic first boundary value problems, we bound the difference between the leftmost terms of (7.18) for u and w. Taking $n \to \infty$ followed by $r \to \infty$ then yields (7.22).

Specifically, note that $\Sigma(u) - \Sigma(w) = (u-w)A_\star$ where

$$A_\star = \int_0^1 A(\tau u + (1-\tau)w)\mathrm{d}\tau$$

is in $C_b(\mathbb{R}_T)$ and $A_\star \geq \underline{a} > 0$. Hence, there exist uniformly bounded smooth functions (A_n, B_n, C_n) such that $A_n \downarrow A_\star$ and $B_n \downarrow B_\star = -hA_\star \in C_b(\mathbb{R}_T)$, uniformly on $\mathbb{R}_T$, whereas m-a.e. $C_n \to C_\star = h_x A_\star$. Now, consider, for each n and $r \geq \ell + 1$, the unique classical solution $f^{(n,r)} \in C^{1,2}(\mathbb{K}_r^o)$ of the first boundary value problem

$$\begin{aligned} \mathscr{L}_n f := f_t + A_n f_{xx} + B_n f_x - C_n f &= 0 \qquad \text{on } (0,t)\times(-r,r), \\ f(t,x) &= \omega(x), \quad x \in [-r,r], \\ f(s,-r) = f(s,r) &= 0, \qquad s \in [0,t]. \end{aligned} \tag{7.23}$$

With A_n bounded away from 0, the existence and uniqueness of such a solution for (7.23) is well-known (see, e.g., [29, chap. IV, theorem 5.2]), and furthermore, with $\omega \geq 0$ also $f^{(n,r)} \geq 0$.

Next, setting $g^{\pm}_{n,r}(s) = [f_x^{(n,r)}(\Sigma(u) - \Sigma(w))](s, \pm r^-)$, we use the test function $f = f^{(n,r)}$ in our modified (7.18) to bound the left-hand side of (7.22) by

$$\begin{aligned} &\int_{-r}^{r} (u(0,x) - w(0,x))_+ f^{(n,r)}(0,x)\mathrm{d}x - \int_0^t \big[g^+_{n,r}(s) - g^-_{n,r}(s)\big]\mathrm{d}s \\ &\quad + \int_{\mathbb{K}_r^o} (u-w)\big[f_{xx}^{(n,r)}(A_\star - A_n) + f_x^{(n,r)}(B_\star - B_n) - f^{(n,r)}(C_\star - C_n)\big]\,\mathrm{d}m. \end{aligned} \tag{7.24}$$

Recall the uniform boundedness of $u - w$ and the uniform convergence to 0 of $A_\star - A_n$ and $B_\star - B_n$. Thus, similarly to the derivation of [11, (4.12)], the last term of (7.24) goes to 0 when $n \to \infty$, provided $\sup_{s,x,n,r} f^{(n,r)}(s,x) \leq c$ finite and both $e_n := \|f_{xx}^{(n,r)}\|_{L^2(\mathbb{K}_r)}$ and $\chi_n := \|f_x^{(n,r)}\|_\infty$ are uniformly bounded in n. Taking then $r \to \infty$ yields (7.22) if in addition $\sup_{s,n} |g^{\pm}_{n,r}(s)| \to 0$ as $r \to \infty$.

Turning to prove the latter four estimates, assume first that m-a.e. $h_x \geq 0$ and hence $C_n \geq 0$ for all n and $(s,x) \in \mathbb{R}_T$. Then, with $A_n \geq \underline{a} > 0$, the maximum principle applies to the parabolic equation (7.23) (see, e.g., [15, sec. 2.1, theorem 1], where our time direction is reversed compared to the setting there), resulting in $0 \leq f^{(n,r)}(s,x) \leq \sup_x \omega(x) \leq 1$ for all n, r (as in [11, lemma 4.1, (i)]). Similarly, taking $\kappa > \sup_n \|A_n + |B_n| - C_n\|_\infty$ and $\mathsf{v}^{(\pm)} = f^{(n,r)} - \phi^{(\pm)}$ for $\phi^{(\pm)} = e^{\ell \pm x + \kappa(t-s)}$, one has that $\mathscr{L}_n \mathsf{v}^{(\pm)} \geq 0$, while $\mathsf{v}^{(\pm)}(s, \pm r) = -\phi^{(\pm)}(s, \pm r) \leq 0$ and $\mathsf{v}^{(\pm)}(t,x) = \omega(x) - \phi^{(\pm)}(t,x) \leq 0$. Hence, by the maximum principle, both $\mathsf{v}^{(+)} \leq 0$ and $\mathsf{v}^{(-)} \leq 0$, yielding the bound $f^{(n,r)}(s,x) \leq e^{\ell - |x|}$ (as in [11, lemma 4.1, (ii)]). Equipped with this bound, we follow the proof of [11, lemma 4.1, (iii)]. Specifically, taking $\xi^{(\pm)} = y(r)e^{\pm\kappa x}$ for $y(r) = e^{\ell - (\kappa+1)(r-1)}$ and constant $\kappa \geq 1$, which makes $\kappa^2 A_n - \kappa|B_n| - C_n$ nonnegative on $\mathbb{R}_T$ for all n, results in $\mathscr{L}_n \mathsf{v}^{(\pm),\pm} \geq 0$ for $\mathsf{v}^{(\pm),\pm} = \xi^{(\pm)} \pm f$ and $f = f^{(n,r)}$. Thus, the four functions $\mathsf{v}^{(\pm),\pm}$ satisfy the maximum principle on each of the two components of $\mathbb{K}_r \backslash \mathbb{K}_{r-1}$.

Since $f(s,\pm r) = f(t,x) = 0$ when $|x| \geq r-1$, while $|f(s,\pm(r-1))| \leq e^{\ell+1-r}$, by our choice of $\xi^{(+)}$, the maximum of $\mathsf{v}^{(+),\pm}$ on the positive component of $\mathbb{K}_r \setminus \mathbb{K}_{r-1}$ is attained at $x = r$, where $\mathsf{v}^{(+),\pm}(s,r) = \xi^{(+)}(r)$ is constant. Hence, $\mathsf{v}_x^{(+),\pm}(s,r^-) \geq 0$, yielding that $|f_x(s,r^-)| \leq \kappa e^{\kappa} e^{\ell+1-r}$. Similarly, the maximum of $\mathsf{v}^{(-),\pm}$ on the negative component of $\mathbb{K}_r \setminus \mathbb{K}_{r-1}$ is attained at $x = -r$, where $\mathsf{v}^{(-),\pm}(s,-r) = \xi^{(-)}(-r)$ is constant. Hence, $\mathsf{v}_x^{(-)}(s,-r^-) \leq 0$, so $|f_x(s,-r^-)| \leq \kappa e^{\kappa} e^{\ell+1-r}$, and $\sup_{s,n} |g_n^{\pm}(s)| \to 0$ when $r \to \infty$. Having uniform ellipticity and (A_n, B_n, C_n) uniformly bounded, the uniform bound on χ_n follows by applying [29, chap. III, theorem 11.1] in our setting. Finally, to bound e_n we multiply the linear PDE (7.23) by f_{xx} and integrate over $\mathbb{K}_r$ as in the proof of [11, lemma 4.1, (v)]. Following the derivation after [11, (4.10)], since $f_t^{(n,r)}(s,\pm r) = 0$, integration by parts of the term $f_t f_{xx}$ results in

$$\begin{aligned} \int_{\mathbb{K}_r^o} A_n \big(f_{xx}^{(n,r)}\big)^2 \, \mathrm{d}m \leq \frac{1}{2} \int_{-r}^{r} \omega'(x)^2 \, \mathrm{d}x \\ + \int_{\mathbb{K}_r^o} f_{xx}^{(n,r)} \big(C_n f^{(n,r)} - B_n f_x^{(n,r)}\big) \mathrm{d}m. \end{aligned} \tag{7.25}$$

Furthermore, $\mathbb{K}_r$ is compact and $C_n f^{(n,r)}$, $B_n f_x^{(n,r)}$ are uniformly bounded. Hence, by the Cauchy-Schwarz inequality, the rightmost term of (7.25) is bounded by $\kappa_2 e_n$ for some $\kappa_2 = \kappa_2(r,h)$ finite. In addition, the left-hand side of (7.25) is at least $\underline{a} e_n^2$, whereas the first term on its right-hand side is some $\kappa_1 = \kappa_1(\omega)$ finite. Consequently, $\underline{a} e_n^2 \leq \kappa_1 + \kappa_2 e_n$, yielding the desired uniform bound on e_n. This completes the proof in case $h_x \geq 0$. More generally, setting $c = e^{\upsilon T}$ for $\upsilon > \|(C_\star)_-\|_\infty$, the function $\widetilde{f}^{(n,r)} = e^{\upsilon(s-t)} f^{(n,r)}$ satisfies (7.23) with $\widetilde{C}_n = C_n + \upsilon \geq 0$. Thus, by the preceding $\sup f^{(n,r)} \leq c \sup \widetilde{f}^{(n,r)} \leq c$, $\sup_n \|g_{n,r}^{\pm}\|_\infty \leq c \sup_n \|\widetilde{g}_{n,r}^{\pm}\|_\infty \to 0$ as $r \to \infty$, and $e_n \leq c \|\widetilde{f}_{xx}^{(n,r)}\|_{L^2(\mathbb{K}_r)}$, $\chi_n \leq c \|\widetilde{f}_x^{(n,r)}\|_\infty$ are both uniformly bounded in n, as claimed. □

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