Problem 6.41

(a) First compute marginal densities \( f_X(x) \) and \( f_Y(y) \). For \( x > 0 \)
\[
f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{0}^{\infty} xe^{-x(y+1)}dy = \left[-e^{-x(y+1)}\right]_0^\infty = e^{-x}
\]
and for \( y > 0 \)
\[
f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{\infty} xe^{-x(y+1)}dx
= \left[-\frac{e^{-x(y+1)}}{y+1} - x\right]_0^\infty + \int_{0}^{\infty} \frac{e^{-x(y+1)}}{y+1}dx
= \left[-\frac{e^{-x(y+1)}}{(y+1)^2}\right]_0^\infty
= \frac{1}{(y+1)^2}
\]
Hence for \( x > 0 \) and \( y > 0 \)
\[
f_{X|Y}(z,x) = \frac{f(x,y)}{|J(x,y)|} = (y+1)^2xe^{-x(y+1)}
\]
and
\[
f_{Y|X}(z) = \frac{f(x,y)}{f_X(x)} = xe^{-xy}
\]
(b) Let \( z = g_1(x,y) = xy \) and \( g_2(x) = x \). Since
\[
J(x,y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x
\]
we get
\[
f_{z,x}(z,x) = f_{X,Y}(x,y)\left|J(x,y)\right|^{-1} = e^{-x(y+1)} = e^{-(z+x)}
\]
Hence
\[
f_z(z) = \int_{0}^{\infty} e^{-(z+x)}dx = e^{-z}
\]
Problem 6.43

Let $A$ denote the number of accidents that a newly insured person will have in her first year and let $X$ denote an accident parameter of this person. Then $A$ has a Poisson distribution with parameter $X$ and $f_X(x) = \frac{se^{-sx}(sx)^{n-1}}{Γ(n)}$ if $x \geq 0$ and $f_X(x) = 0$ otherwise. So

$$P(A = n) = \int_0^\infty P(A = n | X = x) f_X(x) dx = \int_0^\infty e^{-x} x^{n-1} e^{-sx}(sx)^{α-1} \frac{x^{n-1}}{n!} \frac{1}{Γ(α)} dx$$

$$= \frac{s^α Γ(n + α)}{n!(s + 1)^{n+α} Γ(α)} \int_0^\infty (s + 1)e^{-(s+1)x}((s+1)x)^{n+α-1} \frac{x^{n−1}}{Γ(n + α)} dx$$

Hence for $x \geq 0$ we get

$$f_{X|A=n}(x|A=n) = \frac{f_{X,A}(x,n)}{P(A = n)} = \frac{f_{A|X=x}(n|X = x) f_X(x)}{P(A = n)}$$

This means conditional on the event that $A = n$ the accident parameter will have Gamma distribution with parameters $(s + 1, n + α)$. Now, let $Y$ be the number of accidents in the following year. Then

$$E[Y] = \sum_{k=1}^{∞} k P(Y = k) = \sum_{k=1}^{∞} k \int_0^\infty e^{-x} x^k \frac{(s + 1)e^{-(s+1)x}((s+1)x)^{n+α-1}}{Γ(n + α)} dx$$

$$= \int_0^\infty \left( \sum_{k=1}^{∞} e^{-x} x^k \frac{1}{(k-1)!} \right) (s + 1)e^{-(s+1)x}((s+1)x)^{n+α-1} \frac{x^{n−1}}{Γ(n + α)} dx$$

$$= \int_0^\infty x (s + 1)e^{-(s+1)x}((s+1)x)^{n+α-1} \frac{x^{n−1}}{Γ(n + α)} dx$$

$$= \frac{Γ(n + α + 1)}{(s + 1)Γ(n + α)} \int_0^\infty (s + 1)e^{-(s+1)x}((s+1)x)^{n+α} \frac{x^{n−1}}{Γ(n + α + 1)} dx$$

$$= \frac{Γ(n + α + 1)}{(s + 1)Γ(n + α)} \frac{Γ(n + α)}{s + 1}$$

$$= \frac{n + α}{s + 1}$$

Problem 6.45

For $1 \leq i \leq 5$ let $X_i$ denote the length of time that motor $i$ functions. Then

$$P(X_i \leq t) = \int_0^t xe^{-x} dx = \left[-(x + 1)e^{-x}\right]_0^t = 1 - (t + 1)e^{-t}$$

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So if $X$ is the length of the time that machine functions then

$$F_X(t) = P(X \leq t) = \sum_{i=0}^{\lceil t \rceil} \left( \begin{array}{c} 5 \\ i \end{array} \right) (t+1)e^{-t}^i (1-(t+1)e^{-t})^{5-i}$$

Now, take derivative of right hand side to get the density function of $X$.

$$f_X(t) = 30t(t+1)^2e^{-t}((t+1)e^{-t})^2$$

**Problem 6.46**

Let $X_1$, $X_2$ and $X_3$ denote the positions of tracks on a road. Then

$$P(X_1 + 2d < X_2 + d < X_3) = \int_{x_1 + d}^{L-d} \int_{x_2 + d}^{L-d} \int_{x_3 + d}^{L-d} \frac{1}{L^3} dx_3 dx_2 dx_1$$

$$= \frac{1}{L^3} \int_0^{L-2d} \int_{x_1 + d}^{L-d} \int_{x_2 + d}^{L-d} (L-d-x_2) dx_2 dx_1$$

$$= \frac{1}{L^3} \int_0^{L-2d} \left( \frac{(L-d)x_2 - \frac{x_2^2}{2}}{x_1 + d} \right) dx_1$$

$$= \frac{1}{L^3} \int_0^{L-2d} \left( \frac{(L-d)^2}{2} - (L-d)(x_1 + d) + \frac{(x_1 + d)^2}{2} \right) dx_1$$

$$= \frac{1}{L^3} \int_0^{L-2d} \left( \frac{(x_1 - L + 2d)^3}{6} \right) dx_1$$

$$= \frac{1}{6} \left( 1 - \frac{2d}{L} \right)^3$$

We can permute trucks in $3! = 6$ ways. So by symmetry the desired probability is $\left( 1 - \frac{2d}{L} \right)^3$.

**Problem 6.52**

Given that $r = g_1(x, y) = (x^2 + y^2)^{1/2}$ and $\theta = g_2(x, y) = \tan^{-1} y/x$. Since

$$J(x, y) = \left| \begin{array}{cc} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{array} \right| = \left| \begin{array}{cc} x & y \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \end{array} \right| = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

we get

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(x,y)J(x,y)^{-1} = \frac{r}{\pi}$$

for $r \in [0,1)$ and $\theta \in [0, 2\pi)$. 

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Problem 6.T.19

First observe that for distinct \(i, j, k\) we have

\[
P(X_i > X_j > X_k) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_i} \int_{-\infty}^{x_j} f(x_i)f(x_j)f(x_k)dx_kdx_jdx_i
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{x_i} f(x_i)f(x_j)F(x_j)dx_jdx_i
\]

\[
= \int_{-\infty}^{\infty} f(x_i) \left[ \frac{F(x_j)^2}{2} \right]_{-\infty}^{x_i} dx_i
\]

\[
= \int_{-\infty}^{\infty} f(x_i) \frac{F(x_i)^2}{2} dx_i
\]

\[
= \left[ \frac{F(x_i)^3}{6} \right]_{-\infty}^{\infty} = \frac{1}{6}
\]

\[
P(X_i > X_j, X_i > X_k) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_i} \int_{-\infty}^{x_i} f(x_i)f(x_j)f(x_k)dx_kdx_jdx_i
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{x_i} f(x_i)f(x_j)F(x_i)dx_jdx_i
\]

\[
= \int_{-\infty}^{\infty} f(x_i)F(x_i)^2dx_i
\]

\[
= \left[ \frac{F(x_i)^3}{3} \right]_{-\infty}^{\infty} = \frac{1}{3}
\]

\[
P(X_i > X_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_i} f(x_i)f(x_j)dx_jdx_i
\]

\[
= \int_{-\infty}^{\infty} f(x_i)F(x_i)dx_i
\]

\[
= \left[ \frac{F(x_i)^2}{2} \right]_{-\infty}^{\infty} = \frac{1}{2}
\]

(all of these are intuitive by symmetry).

(a)

\[
P(X_1 > X_2|X_1 > X_3) = \frac{P(X_1 > X_2, X_1 > X_3)}{P(X_1 > X_3)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{2}{3}
\]

(b) Since

\[
\frac{1}{2} = P(X_1 > X_2) = P(X_1 > X_2|X_1 > X_3)P(X_1 > X_3) + P(X_1 > X_2|X_1 < X_3)P(X_1 < X_3)
\]

\[
= \frac{2}{3} \cdot \frac{1}{2} + P(X_1 > X_2|X_1 < X_3)\frac{1}{2}
\]
we get \( P(X_1 > X_2 | X_1 < X_3) = \frac{1}{3} \).

(c) 
\[
P(X_1 > X_2 | X_2 > X_3) = \frac{P(X_1 > X_2 > X_3)}{P(X_2 > X_3)} = \frac{1}{\frac{5}{2}} = \frac{2}{5}
\]

(d) Since
\[
\frac{1}{2} = P(X_1 > X_2) = P(X_1 > X_2 | X_2 > X_3)P(X_2 > X_3) + P(X_1 > X_2 | X_2 < X_3)P(X_2 < X_3)
\]
\[
= \frac{1}{2} P(X_1 > X_2 | X_1 < X_3) + P(X_1 > X_2 | X_1 < X_3) \frac{1}{2}
\]
we get \( P(X_1 > X_2 | X_2 < X_3) = \frac{2}{3} \).

**Problem 6.T.20**

(a) Let \( F(t) \) be conditional distribution of \( U \) given that \( U > a \). Then if \( t \leq a \) then
\[
F(t) = P(U \leq t | U > a) = \frac{P(U \leq t, U > a)}{P(U > a)} = 0
\]
and if \( a < t < 1 \) then
\[
F(t) = P(U \leq t | U > a) = \frac{P(a < U \leq t)}{P(U > a)} = \frac{t - a}{1 - a}
\]
and if \( t \geq 1 \) then
\[
F(t) = P(U \leq t | U > a) = \frac{P(a < U \leq t)}{P(U > a)} = \frac{P(a < U)}{P(U > a)} = 1
\]
So \( U \) conditional on \( U > a \) has uniform distribution on \((a, 1)\).

(b) Similarly let \( G(t) \) be conditional distribution of \( U \) given that \( U < a \). Then if \( t \leq 0 \) then
\[
G(t) = P(U \leq t | U < a) = \frac{P(U \leq t, U < a)}{P(U < a)} = 0
\]
and if \( 0 < t < a \) then
\[
G(t) = P(U \leq t | U < a) = \frac{P(U \leq t, U < a)}{P(U < a)} = \frac{P(U \leq t)}{P(U < a)} = \frac{t}{a}
\]
and if \( a \leq t \) then
\[
G(t) = P(U \leq t | U < a) = \frac{P(U \leq t, U < a)}{P(U < a)} = \frac{P(U < a)}{P(U < a)} = 1
\]
So \( U \) conditional on \( U < a \) has uniform distribution on \((0, a)\).
Problem 7.2

(a) Since there are 6 suspects, 6 weapons and 9 rooms there are $6 \cdot 6 \cdot 9 = 324$ possible solutions.

(b) \[ X = (6 - S)(6 - W)(9 - R) \]

(c) Label suspects with $s_i$, weapons with $w_j$ and rooms with $s_k$ for $1 \leq i, j \leq 6$ and $1 \leq k \leq 9$. Let $S$ be the solution. Then by symmetry

\[
E[X] = \sum_{i,j,k} E[X|S = (s_i, w_j, r_k)] P(S = (s_i, w_j, r_k))
\]

\[
= \sum_{i,j,k} E[X|S = (s_i, w_j, r_k)] \frac{1}{324}
\]

\[
= E[X|S = (s_1, w_1, r_1)]
\]

This means it is enough to fix the choice of the first three (the solution) and try to find expectation $E[X]$. Observe that given the solution is $(s_1, w_1, r_1)$ then

\[
(S, W, R) = \begin{cases}
(3, 0, 0) & : \text{with probability } \left(\frac{\binom{5}{3}}{\binom{18}{3}}\right) \\
(2, 0, 1) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
(2, 1, 0) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
(1, 0, 2) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
(1, 1, 1) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
(1, 2, 0) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
(0, 0, 3) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
(0, 1, 2) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
(0, 2, 1) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
(0, 3, 0) & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right)
\end{cases}
\]

So

\[
X = (6 - S)(6 - W)(8 - R) = \begin{cases}
162 & : \text{with probability } \left(\frac{\binom{5}{3}}{\binom{18}{3}}\right) \\
192 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
180 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
210 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
200 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
180 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
216 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
210 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
192 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right) \\
162 & : \text{with probability } \left(\frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{18}{3}}\right)
\end{cases}
\]

Hence

\[
E[X] = \frac{162856}{\binom{18}{3}} = 199.6
\]
Problem 7.8

Let $X_i$ equal 1 or 0 depending on whether the $i$th arrival sits in a previously unoccupied table. Then $X = \sum_{i=1}^{N} X_i$ is the total number of occupied tables. Observe that $X_i = 1$ if neither of the first $i-1$ people is a friend of this person. So $P(X_i = 1) = (1-p)^{i-1}$. Hence by linearity of expectation we get

$$E[X] = \sum_{i=1}^{N} E[X_i] = \sum_{i=1}^{N} P(X_i = 1) = \sum_{i=1}^{N} (1-p)^{i-1} = \begin{cases} N/(1-(1-p)^N) & : p = 0 \\ 1 - (1-p)^N/p & : p \in (0, 1) \end{cases}$$

Problem 7.12

(a) Label men with $M_1, \ldots, M_n$ and women with $W_1, \ldots, W_n$ and let $X_i$ equal 1 or 0 depending on whether the $i$th man sits near a woman. Now let $A_{ij}$ be the event that $i$th man and $j$th woman sit near each other. Observe that there are $2(2n-1)!$ number of outcomes in $A_{ij}$ because we can sit $j$th woman near $i$th man in 2 ways and then thinking these two as one person we can line up $2n-1$ people in $(2n-1)!$ ways. Similarly there are $2(2n-2)!$ outcomes in $A_{ij}A_{ik}$ for $j \neq k$ because we can sit $j$th and $k$th women near $i$th man in 2 ways and then thinking these three as one person we can line up $2n-2$ people in $(2n-2)!$ ways. Thus

$$P(A_{ij}) = \frac{2(2n-1)!}{(2n)!} = \frac{1}{n}$$
and

$$P(A_{ij}A_{ik}) = \frac{2(2n-2)!}{(2n-1)!} = \frac{1}{n(2n-1)}$$
for $j \neq k$. Thus

$$P(X = 1) = P\left(\bigcup_{j=1}^{n} A_{ij}\right) = \sum_{j=1}^{n} P(A_{ij}) - \sum_{j \neq k} P(A_{ij}A_{ik})$$
$$= n \cdot \frac{1}{n} - \frac{n(n-1)}{2} \cdot \frac{1}{n(2n-1)} = \frac{3n-1}{4n-2}$$
Since $X = \sum_{i=1}^{n} X_i$ is the total number of men sitting near a woman by linearity of expectation we get

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P(X_i = 1) = \sum_{i=1}^{n} \frac{3n-1}{4n-2} = \frac{n(3n-1)}{4n-2}$$

(b) This is similar to part (a). Since the are $(k-1)!$ ways to seat $k$ people around table we need to reduce variables in each factorial by 1. So $P(A_{ij}) = \frac{2(2n-2)!}{(2n-1)!} = \frac{2}{2n-1}$ and

$$P(A_{ij}A_{ik}) = \frac{2(2n-3)!}{(2n-1)!} = \frac{1}{(n-1)(2n-1)}$$
for $j \neq k$. Hence

$$P(X = 1) = P\left(\bigcup_{j=1}^{n} A_{ij}\right) = \sum_{j=1}^{n} P(A_{ij}) - \sum_{j \neq k} P(A_{ij}A_{ik})$$
$$= n \cdot \frac{2}{2n-1} - \frac{n(n-1)}{2} \cdot \frac{1}{(n-1)(2n-1)} = \frac{3n}{4n-2}$$

and
\[ E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P(X_i = 1) = \sum_{i=1}^{n} \frac{3n}{4n-2} = \frac{3n^2}{4n-2} \]

**Problem 7.17**

(a) Let \( N_i \) equal 1 or 0 depending on whether the \( i \)th guess is correct. Observe that \( P(N_i = 1) = \frac{1}{n} \). Since \( N = \sum_{i=1}^{n} N_i \) by linearity of expectation we get
\[ E[N] = \sum_{i=1}^{n} E[N_i] = \sum_{i=1}^{n} P(N_i = 1) = \frac{1}{n} = 1 \]

(b) Since we have shown \( i - 1 \) cards before \( i \)th guess in \( i \)th turn we make a guess uniformly out of the rest of \( n - i + 1 \) cards. So \( P(N_i = 1) = \frac{1}{n - i + 1} \). Hence
\[ E[N] = \sum_{i=1}^{n} E[N_i] = \sum_{i=1}^{n} P(N_i = 1) = \frac{1}{n - i + 1} = \frac{1}{n} + \frac{1}{n - 1} + \cdots + 1 \approx \ln n \]

(c) Define \( G_i \) as follows: \( G_i = s \) if you get \( i \)th correct guess in your \( s \)th guess and \( G_i = n + 1 \) if you get less than \( i \) correct guesses. Now observe that if \( G_i = s_i \) then you guessed \( i \) cards correctly and in your \((s_i + 1)\)th guess you select a card uniformly out of the rest of \( n - i \) cards and try this until you get \( G_{i+1} = s_{i+1} \) or until you finish all \( n \) guesses. So
\[ P(G_{i+1} = s_{i+1} | G_i = s_1, \ldots, G_1 = s_1) = P(N_j = 0 \text{ for } j \in [s_i + 1, s_{i+1}] \text{ and } N_{s_{i+1} + 1} = 1) \]
\[ = \frac{n - i - 1}{n - i}, \frac{n - i - 2}{n - i - 1}, \ldots, \frac{n - i - (s_{i+1} - s_i)}{n - i - (s_{i+1} - s_i - 1)}, \frac{1}{n - i} \]
\[ = \frac{1}{n - i} \]
Thus for any \( s_1 < \cdots < s_r \) we get
\[ P(G_1 = s_1, \ldots, G_r = s_r) = P(G_1 = s_1)P(G_2 = s_2 | G_1 = s_1) \cdots P(G_r = s_r | G_{r-1} = s_{r-1}, \ldots, G_1 = s_1) \]
\[ = \frac{1}{n} \cdot \frac{1}{n - 1} \cdots \frac{1}{n - r + 1} = \frac{1}{n(n - 1) \cdots (n - r + 1)} \]
So for any \( 1 \leq r \leq n \) we get
\[ P(N \geq r) = \sum_{s_1 < \cdots < s_r} P(G_1 = s_1, \ldots, G_r = s_r) = \sum_{s_1 < \cdots < s_r} \frac{1}{n(n - 1) \cdots (n - r + 1)} \]
\[ = \frac{1}{n(n - 1) \cdots (n - r + 1)} \binom{n}{r} = \frac{1}{r!} \]
Thus
\[ E[N] = \sum_{r=1}^{\infty} P(N \geq r) = \sum_{r=1}^{n} P(N \geq r) = \sum_{r=1}^{n} \frac{1}{r!} \approx e^{-1} \]