Problem 4.62
Let $E_{ij}^{(\ell)}$ denote the event that trials $i$ and $j$ both have outcome $\ell$ and let $E_{ij} = \bigcup_{\ell=1}^{k} E_{ij}^{(\ell)}$ be the event that trials $i$ and $j$ have the same outcome. Then

$$P(E_{ij}) = \sum_{\ell=1}^{k} P(E_{ij}^{(\ell)}) = \sum_{\ell=1}^{k} p_{\ell}^{2}$$

We know that events $E_{ij}$’s are pairwise independent (problem 4.T.21) and $P(E_{ij}) \leq \max_{1 \leq \ell \leq k} p_{\ell}$ is very small. Since $\sum_{i,j} P(E_{i,j}) = \binom{n}{2} \sum_{\ell=1}^{k} p_{\ell}^{2}$ by Poisson approximation the probability of having no $E_{i}$’s is approximately $\exp \left( -\frac{n(n-1)}{2} \sum_{i=1}^{n} p_{i}^{2} \right)$.

Problem 4.65
(a) Let $A$ be the event that the blood test will be positive. Since all 500 events are independent and each has probability 0.001, which is sufficiently small, we can use Poisson random variable with parameter $500 \cdot 0.001 = 0.5$ to approximate $P(A)$.

$$P(A) \approx 1 - e^{-0.5} = 0.3935$$

(b) Let $B$ be the event that more than one person has the disease. Then

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(B)}{P(A)} \approx \frac{1 - e^{-0.5} - e^{-0.5} \cdot 0.5}{1 - e^{-0.5}} = 0.2293$$

(c) Since Jones know that he has a disease he thinks that probability that more than one person has disease is same as the probability that at least one of remaining 499 soldiers has a disease which is approximately equal to the same probability as in part (a).

(d) Since there are $500 - i$ soldiers to be tested we can use Poisson approximation with parameter $(500 - i) \cdot 0.001$ and deduce that probability that any of the remaining people have the disease is approximately $1 - e^{-0.001(500-i)}$. 


Problem 4.67

(a) Call an arrangement of \( n \) couples around the table good if men and women alternate. To count the number of good arrangements first arrange \( n \) women around table with \((n-1)!\) different ways and then place \( n \) men between these women with \( n! \) different ways. So the sample space of good arrangements has \((n-1)!n!\) outcomes. Now, to count the number of outcomes in \( C_i \) first arrange \( n \) women around table with \((n-1)!\) ways, then place \( i \)'th man near his wife in 2 different ways, depending on the right or left of her, and then place the remaining \( n-1 \) men in remaining seats between women in \((n-1)!\) ways. So there are \(2((n-1)!)^2\) outcomes in \( C_i \). Hence

\[
P(C_i) = \frac{2((n-1)!)^2}{(n-1)!n!} = \frac{2}{n}
\]

(b) Since \( P(C_j | C_i) = \frac{P(C_iC_j)}{P(C_i)} \) we need to count the number of events in \( C_iC_j \). Again first arrange \( n \) women around table in \((n-1)!\) ways. But observe that in \(2(n-2)!\) of these arrangements \( C_i \) and \( C_j \) seat near each other. In any of such arrangements we can place men in \( i \)'th and \( j \)'th couple near their wives in 3 different ways and then place the remaining \( n-2 \) men to the remaining seats between women in \((n-2)!\) ways. So there are \(2(n-2)! \cdot 3 \cdot (n-2)!\) such arrangements. Now, in remaining \((n-1)! - 2(n-2)! = (n-3)(n-2)!\) arrangements \( C_i \) and \( C_j \) do not seat near each other. So in all of such arrangements men in \( i \)'th and \( j \)'th couple can be seated near their wives in 4 different ways and then we can place the remaining \( n-2 \) men to the remaining seats between women in \((n-2)!\) ways. So there are \((n-3)(n-2)! \cdot 4 \cdot (n-2)!\) such arrangements. Thus \( C_iC_j \) has \(2(n-2)! \cdot 3 \cdot (n-2)! + (n-3)(n-2)! \cdot 4 \cdot (n-2)! = 2(2n-3)((n-2)!)^2\) outcomes. So

\[
P(C_iC_j) = \frac{2(2n-3)((n-2)!)^2}{(n-1)!n!} = \frac{2(2n-3)}{n(n-1)^2}
\]

and

\[
P(C_j | C_i) = \frac{2(2n-3)}{n(n-1)^2} = \frac{2n - 3}{(n-1)^2}
\]

(c) Let \( C \) denote the event that there are no couple seated near each other. Observe that each of \( C_i \) has small probability and dependency between them is “weak”. So we can use Poisson random variable with parameter \( n \cdot \frac{2}{n} = 2 \) to deduce that \( P(C) \) is approximately \( e^{-2} \).

Problem 4.69

Let \( A \) be the event that there is a string of 4 consecutive heads
(a) Using equation on page 143 with \( n = 10 \) and \( k = 4 \) we get

\[
P(A) = \sum_{r=1}^{7} (-1)^{r+1} \left[ \binom{10 - 4r}{r} + 2 \binom{10 - 4r}{r - 1} \right] \left( \frac{1}{2} \right)^{5r}
\]

\[
= \left[ \binom{6}{1} + 2 \binom{6}{0} \right] \left( \frac{1}{2} \right)^{5} - \left[ \binom{2}{2} + 2 \binom{2}{1} \right] \left( \frac{1}{2} \right)^{10}
\]

\[
= \frac{251}{1024} \approx 0.2451
\]

(b) Use recursive equation on page 143 with \( n = 10 \) and \( k = 4 \). Obviously \( P_1 = P_2 = P_3 = 0 \) and

\[
P_4 = \left( \frac{1}{2} \right)^{4} = \frac{1}{16}
\]

Then

\[
P_5 = P_4(1/2) + (1/2)^4 = 3/32
\]

\[
P_6 = P_5(1/2) + P_4(1/2)^2 + (1/2)^4 = 8/64
\]

\[
P_7 = P_6(1/2) + P_5(1/2)^2 + P_4(1/2)^3 + (1/2)^4 = 20/128
\]

\[
P_8 = P_7(1/2) + P_6(1/2)^2 + P_5(1/2)^3 + P_4(1/2)^4 + (1/2)^4 = 48/256
\]

\[
P_9 = P_8(1/2) + P_7(1/2)^2 + P_6(1/2)^3 + P_5(1/2)^4 + (1/2)^4 = 111/512
\]

\[
P_{10} = P_9(1/2) + P_8(1/2)^2 + P_7(1/2)^3 + P_6(1/2)^4 + (1/2)^4 = 251/1024 \approx 0.2451
\]

(c) We use Poisson distribution with parameter \((10 - 4) \left( \frac{1}{2} \right)^{4} \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} \right)^{4} = \frac{1}{4}\) and get

value \( 1 - \exp \left( -\frac{1}{4} \right) \approx 0.2212 \).

**Problem 4.72**

Let \( X \) be the number of games to be played so that the first team wins the game. Then \( X \) has a negative binomial distribution with \( r = 4 \) and \( p = 0.6 \). So

\[
P(X = i) = \binom{i - 1}{3} \cdot 0.6^4 \cdot 0.4^{i-4}
\]

\[
P(X = 4) = \frac{3}{3} \cdot 0.6^4 \cdot 0.4^0 = 0.6^4
\]

\[
P(X = 5) = \frac{4}{3} \cdot 0.6^4 \cdot 0.4^1 = 1.6 \cdot 0.6^4
\]

\[
P(X = 6) = \frac{5}{3} \cdot 0.6^4 \cdot 0.4^2 = 1.6 \cdot 0.6^4
\]

\[
P(X = 7) = \frac{6}{3} \cdot 0.6^4 \cdot 0.4^3 = 1.28 \cdot 0.6^4
\]
Then the probability that the stronger team wins is \( P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) \approx 0.710 \). On the other hand, the probability that it would win 2-out-of-3 series is 
\[
\binom{3}{2}0.6^2 \cdot 0.4 + \binom{3}{3}0.6^3 = 0.648
\]
which is less than the probability that the stronger team wins.

**Problem 4.76**

Let \( E_r \) denote the event that the mathematician first discovers that the right-hand box is empty and there are \( k \) matches in the left-hand box at the time. This means \((N_2 + 1)\)’th choice from right-hand box is made at \((N_2 + 1 + N_1 - k)\)’th trial. So

\[
P(E_r) = \left( \frac{N_1 + N_2 - k}{N_2} \right) \left( \frac{1}{2} \right)^{N_1 + N_2 - k + 1}
\]

Similarly, if \( E_l \) denote the event that the mathematician first discovers that the left-hand box is empty and there are \( k \) matches in the right-hand box at the time, then

\[
P(E_l) = \left( \frac{N_1 + N_2 - k}{N_1} \right) \left( \frac{1}{2} \right)^{N_1 + N_2 - k + 1}
\]

So desired probability is equal to

\[
P(E_r) + P(E_l) = \left[ \left( \frac{N_1 + N_2 - k}{N_2} \right) + \left( \frac{N_1 + N_2 - k}{N_1} \right) \right] \left( \frac{1}{2} \right)^{N_1 + N_2 - k + 1}
\]

**Problem 4.T.25**

Let \( X \) be the number of events that occur in a specified time. Then \( P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \). Now, if \( Y \) is the number of events that are counted, then by Bayes’s formula

\[
P(Y = j) = \sum_{k=1}^{\infty} P(Y = j | X = k) P(X = k) = \sum_{k=j}^{\infty} \left[ \binom{k}{j} p^j (1 - p)^{k-j} \right] e^{-\lambda} \frac{\lambda^k}{k!}
\]

\[
= e^{-\lambda} \frac{(\lambda p)^j}{j!} \sum_{k=j}^{\infty} \frac{(\lambda(1 - p))^{k-j}}{(k-j)!} = e^{-\lambda} \frac{(\lambda p)^j}{j!} \cdot e^{\lambda (1 - p)}
\]

\[
= e^{-\lambda p} \frac{(\lambda p)^j}{j!}
\]

So \( Y \) is a Poisson random variable with parameter \( \lambda p \).

Since the number of discovered deposits is Poisson with parameter \( 10 \cdot \frac{1}{50} = 0.2 \)

(a) \( P(\text{exactly 1 deposit is discovered}) = e^{-0.2} \)

(b) \( P(\text{at least 1 deposit is discovered}) = 1 - P(\text{no deposit is discovered}) = 1 - e^{-0.2} \)

(c) \( P(\text{at most 1 deposit is discovered}) = e^{-0.2} + e^{-0.2} = 2e^{-0.2} \)
Problem 4.T.28

First solution: Need to prove that

\[ \sum_{i=n+1}^{\infty} \binom{i-1}{r-1} p^r (1-p)^{i-r} = \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i} \]

Denote LHS by \( L_n \) and RHS by \( R_n \) and use induction on \( n \). If \( n < r-1 \) then \( L_n = R_n = 0 \). If \( n = r - 1 \) then \( L_n = R_n = 1 \). Now suppose that \( L_n = R_n \) and prove for \( n + 1 \)

\[ L_{n+1} = \sum_{i=n+2}^{\infty} \binom{i-1}{r-1} p^r (1-p)^{i-r} = \sum_{i=n+1}^{\infty} \binom{i-1}{r-1} p^r (1-p)^{i-r} - \binom{n}{r-1} p^r (1-p)^{n+1-r} \]

Now use identity \( \binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1} \) to compute

\[ R_{n+1} = \sum_{i=0}^{r-1} \binom{n+1}{i} p^i (1-p)^{n+1-i} = \sum_{i=0}^{r-1} \left[ \binom{n}{i} + \binom{n}{i-1} \right] p^i (1-p)^{n+1-i} \]

\[ = \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n+1-i} + \sum_{i=1}^{r-1} \binom{n}{i-1} p^i (1-p)^{n+1-i} \]

\[ = (1-p) \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i} + p \sum_{i=1}^{r-1} \binom{n}{i-1} p^{i-1} (1-p)^{n-(i-1)} \]

\[ = (1-p) R_n + p R_n - \binom{n}{r-1} p^r (1-p)^{n+1-r} \]

\[ = R_n - \binom{n}{r-1} p^r (1-p)^{n+1-r} \]

Since \( L_n = R_n \) we conclude that \( L_{n+1} = R_{n+1} \).

Second solution: Since \( X \) is the number of the trial where we observe the \( r \)’th head event \( \{ X > n \} \) is the collection of outcomes which has less than \( r \) heads in the first \( n \) trials. Now, since \( Y \) counts the number of heads in the first \( n \) trials event \( \{ Y < r \} \) is the collection of all outcomes in which there are less than \( r \) heads in \( n \) trials. Thus \( \{ X > n \} \) and \( \{ Y < r \} \) represent the same collection of outcomes. So \( P(X > n) = P(Y < r) \).

Problem 4.T.32

Obviously \( X \) takes integer values from \( \{2, 3, \ldots, n + 1\} \). Event \( X = k \) means that the first \( k - 1 \) chosen chips are all distinct and \( k \)’th chosen chip is one of the first \( k - 1 \) chosen chips. Since there are \( n(n-1) \cdots (n-k+2) \) ways to choose the first \( k - 1 \) chips and \( k - 1 \) ways to choose \( k \)’th chip we deduce there are \( n(n-1) \cdots (n-k+2)(k-1) \) outcomes in \( X = k \). Hence

\[ P(X = k) = \frac{n(n-1) \cdots (n-k+2)(k-1)}{n^k} \]
Problem 4.T.35

(a) Let $Y_k$ be the event that $k$’th selection is red. Then

\[ P(X > i) = P\left( \bigcap_{k=1}^{i} Y_k \right) = \prod_{k=1}^{i} P(Y_k | Y_1 Y_2 \ldots Y_{k-1}) = \prod_{k=1}^{i} \frac{k}{k+1} = \frac{1}{i+1} \]

(b) Since events $\{X \leq i\}$ are increasing in $i$ by Proposition 2.6.1, we get

\[ P(X < \infty) = P\left( \bigcup_{i=1}^{\infty} \{ X \leq i \} \right) = \lim_{i \to \infty} P(X \leq i) = \lim_{i \to \infty} \frac{i}{i+1} = 1 \]

(c) Since

\[ P(X = i) = P(X > i - 1) - P(X > i) = \frac{1}{i} - \frac{1}{i+1} = \frac{1}{i(i+1)} \]

we get

\[ E[X] = \sum_{i=1}^{\infty} i P(X = i) = \sum_{i=1}^{\infty} \frac{1}{i+1} = \infty \]