1. Exercise 1.1.3. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(A, B, A_i\) events in \(\mathcal{F}\). Prove the following properties of \(\mathbb{P}\).

(a) Monotonicity. If \(A \subseteq B\) then \(\mathbb{P}(A) \leq \mathbb{P}(B)\).

**ANS:** \(A \subseteq B\) implies that \(B = A \cup (B\setminus A)\). Hence, \(\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B\setminus A)\). Thus since \(\mathbb{P}(B\setminus A) \geq 0\), we get \(\mathbb{P}(A) \leq \mathbb{P}(B)\).

(b) Subadditivity. If \(A \subseteq \bigcup_i A_i\) then \(\mathbb{P}(A) \leq \sum_i \mathbb{P}(A_i)\).

**ANS:** For each \(i\) set \(B_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j\). Then the \(B_i\) are disjoint and we let \(C = \bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i\). Since \(A \subseteq C\), from part (a), \(\mathbb{P}(A) \leq \mathbb{P}(C)\). Also, \(\mathbb{P}(C) = \sum_{i=1}^\infty \mathbb{P}(B_i)\) and \(B_i \subseteq A_i\) therefore \(\mathbb{P}(B_i) \leq \mathbb{P}(A_i)\) so \(\mathbb{P}(C) \leq \sum_{i=1}^\infty \mathbb{P}(A_i)\) and hence \(\mathbb{P}(A) \leq \sum_{i=1}^\infty \mathbb{P}(A_i)\).

(c) Continuity from below: If \(A_i \uparrow A\), that is, \(A_1 \subseteq A_2 \subseteq \ldots \) and \(\bigcup_i A_i = A\), then \(\mathbb{P}(A_i) \uparrow \mathbb{P}(A)\).

**ANS:** Construct the disjoint sets \(B_1 = A_1\) and \(B_i = A_i \setminus A_{i-1}\) for \(i \geq 2\), noting that \(A_i = \bigcup_{j \leq i} B_j\) and \(A = \bigcup_i B_i\). Therefore, \(\mathbb{P}(A_i) = \sum_{j=1}^i \mathbb{P}(B_j) \uparrow \sum_{j=1}^\infty \mathbb{P}(B_j) = \mathbb{P}(\bigcup_i B_i) = \mathbb{P}(A)\).

(d) Continuity from above: If \(A_i \downarrow A\), that is, \(A_1 \supseteq A_2 \supseteq \ldots \) and \(\bigcap_i A_i = A\), then \(\mathbb{P}(A_i) \downarrow \mathbb{P}(A)\).

**ANS:** Apply part (c) to the sets \(A_i^c \uparrow A^c\) to have that \(1 - \mathbb{P}(A_i) = \mathbb{P}(A_i^c) \uparrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)\).

(e) Inclusion-exclusion rule:

\[
\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap \cdots \cap A_n).
\]

**ANS:** The proof is by induction on \(n\). The case where \(n = 1\) is immediate. For \(n = 2\), we observe

\[
\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1 \cup [A_2 \setminus (A_1 \cap A_2)]) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).
\]

Suppose the result holds for some \(n \geq 2\). Applying the result to the two sets \(\bigcup_{i=1}^n A_i\) and \(A_{n+1}\), we see

\[
\mathbb{P}(A_1 \cup \cdots \cup A_{n+1}) = \mathbb{P}(A_1 \cup \cdots \cup A_n) + \mathbb{P}(A_{n+1}) - \mathbb{P}((A_1 \cap A_{n+1}) \cup \cdots \cup (A_n \cap A_{n+1})).
\]

Inclusion-exclusion for \(n + 1\) now follows by applying the case for \(n\) to the first and last probabilities on the right hand side and rearranging.
2. **Exercise 1.1.9**. Verify the alternative definitions of the Borel σ-field \( \mathcal{B} \):

\[
\sigma(\{(a,b) : a < b \in \mathbb{R}\}) = \sigma(\{(a,b) : a < b \in \mathbb{R}\}) = \sigma(\{(-\infty,b) : b \in \mathbb{R}\}) = \sigma(\{O \subseteq \mathbb{R} \text{ open}\})
\]

Hint: Any \( O \subseteq \mathbb{R} \) open is a countable union of sets \((a,b)\) for \( a, b \in \mathbb{Q} \) (rational).

**ANS:** Let \( \sigma_1 = \sigma(\{(a,b) : a < b \in \mathbb{R}\}) \), \( \sigma_2 = \sigma(\{[a,b] : a < b \in \mathbb{R}\}) \), \( \sigma_3 = \sigma(\{(-\infty,b) : b \in \mathbb{R}\}) \), \( \sigma_4 = \sigma(\{(-\infty,b) : b \in \mathbb{Q}\}) \) and \( \sigma_5 = \sigma(\{O \subseteq \mathbb{R} \text{ open}\}) \), be the five σ-fields appearing in the problem. Recall that if a collection of sets \( \mathcal{A} \) is a subset of a σ-field \( \mathcal{F} \), then also \( \mathcal{A} \subseteq \mathcal{F} \). For this reason we have that \( \sigma_1 \subseteq \sigma_5 \) and defining \( \sigma_0 = \sigma(\{(a,b) : a < b \in \mathbb{Q}\}) \), we have for same reason that \( \sigma_0 \subseteq \sigma_1 \). By the hint provided we see that any open set \( O \) is a countable union of sets in \( \sigma_0 \), hence also in \( \sigma_0 \). Therefore, \( \sigma_5 \subseteq \sigma_0 \), forcing in view of the above \( \sigma_0 = \sigma_1 = \sigma_5 \). Since \(-\infty,b\) is the countable union of \([b-i,i], i = 1,2,\ldots\), it follows that \(-\infty,b\) \( \in \sigma_2 \) for any \( b \in \mathbb{R} \), hence \( \sigma_4 \subseteq \sigma_3 \subseteq \sigma_2 \). Since each set \([a,b]\) can be expressed as the countable intersection \( \bigcap_{i=1}^{\infty} (-1/i, b+1/i) \), we see that \( \sigma_3 \subseteq \sigma_1 \). Further, since \([b,\infty)\) is the countable intersection of the complements of \((-\infty,b-i]\), \( i = 1,2,\ldots \), it follows that \([b,\infty) \in \sigma_4 \) for \( b \in \mathbb{Q} \), hence \((a,b)\) which is the complement of the union of \((-\infty,a]\) and \([b,\infty)\) is in \( \sigma_4 \) when \( a, b \in \mathbb{Q} \), resulting with \( \sigma_0 \subseteq \sigma_4 \). Recall we have shown that \( \sigma_0 = \sigma_1 = \sigma_5 \) and just now saw that \( \sigma_0 \subseteq \sigma_4 \subseteq \sigma_3 \subseteq \sigma_2 \subseteq \sigma_1 \), implying all six σ-fields considered are the same.

3. **Exercise 1.1.12** Check that the following are Borel sets and find the probability assigned to each by the uniform measure from Example 1.1.11: \((0,1/2) \cup (1/2,3/2), \{1/2\}\), a countable subset \( A \) of \( \mathbb{R} \), the set of irrational numbers in \((0,1), [0,1], \) and \( \mathbb{R} \).

**ANS:** \((0,1/2) \cup (1/2,3/2)\) is open and hence Borel. By countable additivity,

\[
U((0,1/2) \cup (1/2,3/2)) = U((0,1/2)) + U((1/2,3/2)) = 1/2 + 1/2 = 1.
\]

The singleton \( \{1/2\}\) is closed and hence Borel. There are two easy ways to see that \( U(\{1/2\}) = 0 \). First, fixing \( \epsilon > 0 \) arbitrary, we see that

\[
U(\{1/2\}) \leq U((1/2 - \epsilon/2,1/2 + \epsilon/2)) = \epsilon.
\]

Second,

\[
1 = U((0,1)) = U((0,1/2) \cup (1/2,1) \cup \{1/2\}) = 1/2 + 1/2 + U(\{1/2\}).
\]

If \( A \subseteq \mathbb{R} \) is countable then we can write \( A = \bigcup_{n=1}^{\infty} \{a_n\} \) for \( a_n \in \mathbb{R} \). Since each \( \{a_n\} \) is closed, \( A \) is a countable union of closed sets and hence Borel. Either \( a_n \in (0,1) \) or \( a_n \notin (0,1) \). In the former case we
can argue as before to get that $U(\{a_n\}) = 0$ and in the latter case that $U(\{a_n\}) = 0$ is trivial. Hence by countable subadditivity,
\[ U(A) \leq \sum_{n=1}^{\infty} U(\{a_n\}) = 0. \]

Let $J$ denote the set of rationals in $(0, 1)$. Then $J$ is countable and hence Borel with $U(J) = 0$. The set $I$ of irrationals in $(0, 1)$ is thus Borel since we can write $I = (0, 1) \setminus J$. We have,
\[ U(I) = U((0, 1) \setminus J) = U((0, 1)) - U(J) = 1. \]

The set $[0, 1]$ is Borel since it is closed. We have,
\[ U([0, 1]) = U((0, 1)) + U(\{0\}) + U(\{1\}) = 1. \]

Finally, the set of reals $\mathbb{R}$ is Borel since it is open. We have,
\[ U(\mathbb{R}) = U(\mathbb{R} \cap (0, 1)) = U((0, 1)) = 1. \]

4. **Exercise 1.2.5**. Let $\Omega = \{1, 2, 3\}$. Find a $\sigma$-field $\mathcal{F}$ such that $(\Omega, \mathcal{F})$ is a measurable space, and a mapping $X$ from $\Omega$ to $\mathbb{R}$, such that $X$ is not a random variable on $(\Omega, \mathcal{F})$.

**ANS:** Let $\mathcal{F} = \sigma(\{1, 2, 3\}) = \{\{1, 2, 3\}, \emptyset\}$ be the trivial $\sigma$-field. Together $(\Omega, \mathcal{F})$ form a measurable space. Let $X(\omega) = \omega$ where $\omega \in \Omega$. Then $\{\omega : X(\omega) \leq 1\} = \{1\} \notin \mathcal{F}$, so $X$ is not a random variable.

5. **Exercise 1.2.18**. Provide an example of a measurable space, a R.V. on it, and:
   (a) A function $g(x) \neq x$ such that $\sigma(g(X)) = \sigma(X)$.

   **ANS:** Take $\Omega = \mathbb{R}$, $\mathcal{B}$ the Borel sets on $\mathbb{R}$, $X(x) = x$, and $g(x) = -x$. Then $\sigma(X) = \sigma(-X) = \mathcal{B}$.
   
   (b) A function $f$ such that $\sigma(f(X))$ is strictly smaller than $\sigma(X)$ and is not the trivial $\sigma$-field $\{\emptyset, \Omega\}$.

   **ANS:** Take $\Omega, \mathcal{B}$, and $X$ as before and set $f(x) = 1_{(0,1)}(x)$. Then $\sigma(X) = \mathcal{B}$ but $\sigma(f(X)) = \sigma((0,1)) = \{\emptyset, \mathbb{R}, (0,1), (0,1)^c\} \neq \mathcal{B}$.

6. **Exercise 1.2.40**. Show that if $\mathbb{E}[X^2] = 0$ then $X = 0$ almost surely.

   **ANS:** For $n \in \mathbb{N}$, let $A_n = \{|X| > 1/n\}$. Note that $\{X \neq 0\} = \cup_n A_n$. Hence by countable subadditivity it suffices to show that $\mathbb{P}(A_n) = 0$ for all $n$. This follows immediately by applying Markov’s inequality (Theorem 1.2.38) to the function $f(x) = x^2$:
\[ \mathbb{P}(A_n) \leq n^2\mathbb{E}[X^2] = 0. \]