1. (3x14) State which of the following statements is true and which is false. You get 1 point for each correct answer (-1 point for each wrong answer) +2 extra points for the correct reasoning (that is, citing lecture notes, deriving from known result or providing a counter example).

a) For any real-valued random variable $X$, its distribution function uniquely determines its law.

True. This is Proposition 1.4.6.

b) The sample path of the Brownian motion is right-continuous with probability one.

True. By definition the sample path of the Brownian motion is continuous with probability one (property (c) of Definition 5.1.1), and continuity implies right-continuity.

c) The collection of R.V. $\{W_t, t \geq 0\}$ is a uniformly integrable martingale (where $W_t$ denotes the Brownian motion).

False. The Brownian motion is a continuous (hence separable) martingale. If the statement was true then by Doob’s convergence theorem $W_t$ would converge to some random variable $W_\infty$ (almost surely and) in $L^1$. In particular, we would have that $E|W_t| \to E|W_\infty| < \infty$. However, $W_t$ has the same law as $\sqrt{t}W_1$, so $E|W_t| = \sqrt{t}E|W_1| \to \infty$ as $t \to \infty$, a contradiction (see also the discussion following Corollary 4.5.2).

d) If $\sigma(X) = \sigma(Y)$ for two random variables $X$ and $Y$ on the same probability space, then $X = Y$ almost surely.

False. For example take $Y = X + 1$. Since $Y$ is a non-random invertible function of $X$ we know that $\sigma(Y) = \sigma(X)$ (see Corollary 1.2.17). But $X(\omega)$ never equals $Y(\omega)$.

e) If integrable random variables $X_n$ are such that $X_n \to 0$ almost surely, then $E(X_n) \to 0$ as well.

False. We saw many sequences $X_n$ that converge to zero almost surely, while $E(X_n) = 1$ for all $n$. One such example is the non-trivial critical branching process (see the remark following Proposition 4.6.5).

f) There exists a random variable $X \in L^1$ and a $\sigma$-field $\mathcal{G}$, such that the $L^1$ norm of $X$ is smaller than the $L^1$ norm of $Y = E(X|\mathcal{G})$.

False. We saw that always $E|X| \geq E(|E(X|\mathcal{G})|$ as an application of Jensen’s inequality (Corollary 2.3.11).

g) Knowing all finite dimensional distributions of a stochastic process $X_t, 0 \leq t \leq 1$, uniquely determines the law of $\sup_{0 \leq t \leq 1} X_t$.

False. For example the stochastic processes $Y_t = 0$ and $X_t(\omega) = 1_{\{t=\omega\}}$ have the same f.d.d. but $\sup_t X_t = 1$ almost surely, while $\sup_t Y_t = 0$ almost surely (this is Example 3.1.7).

h) In the critical case (that is, when the mean number of children is 1), the Branching process $Z_n$ is a non-negative martingale.

True. We saw that $m^{-n}Z_n$ is a non-negative martingale (Proposition 4.6.2 and the reason we studied branching processes), where $m = 1$ in the critical case.
i) If for any pair of times $t$ and $s$, the covariance $\mathbf{E}(X_tX_s)$ of a zero-mean Gaussian process $X_t$ depends only on the time difference $t - s$, then $X_t$ is a stationary stochastic process.

**True.** The f.d.d. of Gaussian processes are completely determined by their mean and auto-correlation functions, leading to the above simple criterion for stationarity (see Proposition 3.2.25).

j) Any version $Y_t$ of a given stochastic process $X_t$, defined on the same probability space, is also a modification of $X_t$.

**False.** A counter example is given in Example 3.1.11 (see Exercise 3.1.12).

k) If $X_t$ is a sub-martingale for a filtration $\mathcal{F}_t$, then $-X_t$ is a super-martingale for the same filtration.

**True.** This is immediate consequence of the definitions of sub and super martingale (see Remark 4.1.19).

l) For any continuous time stochastic process $X_t$, the collection $\sigma(X_s, 0 \leq s \leq t)$ is a filtration.

**True.** This is what we call the canonical filtration and use alot when dealing with martingales (see the remark following Definition 4.2.1).

m) Suppose $X_i$ are independent, non-negative random variables defined on the same probability space, with $\mathbf{E}(X_i) = 1$ for each $i$. Then, $M_n = \prod_{i=1}^{n} X_i$ is a martingale with respect to $\sigma(X_i, i = 1, \ldots, n)$.

**True.** This is the martingale of Example 4.1.14.

n) Any two continuous-time stochastic processes with stationary independent increments have the same finite dimensional distributions.

**False.** The Brownian motion $W_t$ has independent increments (Proposition 5.1.2), which are stationary, that is the law of $W_{t+h} - W_t$ depends only on $h$. The same applies to $cW_t$ for any $c \neq 0$, which has different f.d.d. when $c \neq 1$ (alternatively, the Poisson process $N_t$ also has stationary, independent increments, see Definition 6.2.1).

2. (14) $T_i, i = 1, 2, \ldots$ are independent Exponential($\lambda$) random variables. Let $M_n = \max_{1 \leq i \leq n} \{ T_i \}$. Find non-random numbers $a_n$ and a non-zero random variable $M_{\infty}$ such that $(M_n - a_n)$ converges in law (distribution), to $M_{\infty}$.

**Solution:** $a_n = \lambda^{-1} \log n$ and the distribution function of $M_{\infty}$ is $F_{M_{\infty}}(x) = \exp(-e^{-\lambda x})$ (this function is monotone increasing from 0 to 1 and differentiable everywhere, hence a distribution function of a R.V. with density). Indeed, since $M_n$ is the maximum of $n$ I.I.D. random variables $T_i$, each of which having the distribution function $F_T(t) = 1 - e^{-\lambda t}$ for $t \in [0, \infty)$, we have that

$$
P(M_n \leq \lambda^{-1} \log n + x) = \prod_{i=1}^{n} P(T_i \leq \lambda^{-1} \log n + x) = (1 - n^{-1} e^{-\lambda x})^n,$$

for all $x \geq -\lambda^{-1} \log n$. Fixing any real-valued $x$, in the limit $n \to \infty$ we thus get that

$$
P(M_n \leq \lambda^{-1} \log n + x) \to \exp(-e^{-\lambda x}).$$

This amounts to $(M_n - a_n)$ converging in law to $M_{\infty}$ (see Definition 1.4.10).

3. (6+6+6) The random variables $X, Y$ and $Z$ are defined on the same probability space. Give as explicit as possible expression for $\mathbf{E}(X|Y)$, in the following three cases. Justify your answers.

a) $X = f(Y) + Z$ for some non-random, integrable Borel function $f$ and a Poisson(1) random variable $Z$ which is independent of $Y$. 

Solution: By linearity of the conditional expectation $E(X|Y) = E(f(Y)|Y) + E(Z|Y)$ (Proposition 2.3.4). Since $Z$ is independent of $Y$ we have that $E(Z|Y) = E(Z)$ (Example 2.3.1), while with $f(Y) \in \sigma(Y)$ we have that $E(f(Y)|Y) = f(Y)$ (Example 2.3.2). Finally, $Z$ is Poisson(1) variable, hence of mean one. In conclusion $E(X|Y) = f(Y) + 1$.

b) $X = W_t$ and $Y = 1_{W_s>0}$, where $t > s$ and $W_t$ is a Brownian motion.

Solution: Write $X = (W_t - W_s) + W_s$ so by linearity of the C.E. $E(X|Y) = E(W_t - W_s|Y) + E(W_s|Y)$. Note that $Y \in \mathcal{F}_s = \sigma(W_u, 0 \leq u \leq s)$, so $\sigma(Y) \subseteq \mathcal{F}_s$ and by the tower property $E(W_t - W_s|Y) = E(E(W_t - W_s|\mathcal{F}_s)|Y) = 0$ (the latter since Brownian motion is a process of zero-mean independent increments). We realize the symmetric Gaussian random variable $W_s$ by choosing first $Y \in \{0, 1\}$ with $P(Y = 1) = 1/2$, then $W_s = (-1)^Y Z$ with $Z \leq 0$ independent of $Y$ and having the density $f_Z(z) = 2(2\pi)^{-1/2}e^{-z^2/(2s)}1_{z \leq 0}$. It follows that $E(W_s|Y) = (-1)^Y E(Z)$. Moreover, it is not hard to check that $E(Z) = \int z f_Z(z) dz = -\sqrt{2s/\pi}$. In conclusion, $E(X|Y) = (-1)^Y \sqrt{2s/\pi}$.

c) $X$ and $Z = h(Y)$ are square integrable and $E[Xg(Y)] = E[Zg(Y)]$ for any Borel function $g$ such that $E(g(Y))^2 < \infty$.

Solution: Let $\mathcal{G} = \sigma(Y)$. Note that $Z \in L^2(\Omega, \mathcal{G}, P)$ is by the statement above such that $E((X - Z) V) = 0$ for all $V \in L^2(\Omega, \mathcal{G}, P)$ (see the structure of $\mathcal{G}$ in Theorem 1.2.14). This is exactly Proposition 2.1.2 about conditional expectation via orthogonal projection. That is, $E(X|Y) = E(X|\mathcal{G}) = Z = h(Y)$.

4. (8+8) This problem is taken from your homework. You should solve it again (and not merely cite the homework solution).

a) Find a non-random $f(t)$ such that $X_t = e^{W_t-f(t)}$ is a martingale.

Solution: $f(t) = t/2$. Here is why. Let $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)$, noting that $X_t$ is adapted to $\mathcal{F}_t$. Moreover, $X_{t+h} = X_te^{W_{t+h} - W_t - h/2}$ for all $t, h > 0$. Let $Z_h^{(t)} = e^{W_{t+h} - W_t - h/2}$ which is independent of $\mathcal{F}_t$ and has the law of $e^{W_t - h/2}$. Since $W_t$ is zero-mean Gaussian of variance $h$, it is not hard to check that $E(Z_h^{(t)}|\mathcal{F}_t) = E(e^{W_t - h/2}) = 1$ (use the Gaussian density $f_{W_t}(w)$ noting that $e^{w - h/2} f_{W_t}(w) = f_{W_t}(w - h)$). Therefore, the non-negative $X_t$ is integrable, and moreover, $(X_t, \mathcal{F}_t)$ is also a martingale.

b) For this value of $f(t)$ find the increasing part of the Doob-Meyer decomposition for the martingale $X_t$ (Hint: it should be of the form $\int_0^t e^{2W_s-h(s)}ds$ for some non-random $h(s)$).

Solution: Since $2W_t$ has the same law as $W_{4t}$ it follows from the above that $E(e^{2W_{h} - h}) = h$ for all $h \geq 0$. Let $G_h^{(t)} = e^{2(W_{t+h} - W_t) - h}$ which is independent of $\mathcal{F}_t$ and has the law of $e^{2W_t - h}$. Then, $E(G_h^{(t)} - 1 - \int_0^t G_u^{(t)} du|\mathcal{F}_t) = 0$ for all $h \geq 0$, implying for $f(t) = t/2$ that $X_t^2 - \int_0^t e^{2W_s - s}ds$ is a martingale for $\mathcal{F}_t$. The sample path continuity of $W_t$ implies the same for $X_t$ and $E(X_t^2) < \infty$, so by the uniqueness of the increasing process (see Doob-Meyer decomposition in Section 4.4), it must be $A_t = \int_0^t e^{2W_s - s}ds$. Check that this process $A_t$ satisfies the five properties of Doob-Meyer decomposition.